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Physics Reports 352 (2001) 439–458

PHYSICS REPORTS

www.elsevier.com/locate/physrep

Renormalization group and probability theory

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Received March 2001; editor: I. Procaccia

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Abstract

The renormalization group has played an important role in the physics of the second half of the 20th century both as a conceptual and a calculational tool. In particular, it provided the key ideas for the construction of a qualitative and quantitative theory of the critical point in phase transitions and started a new era in statistical mechanics. Probability theory lies at the foundation of this branch of physics and the renormalization group has an interesting probabilistic interpretation as it was recognized in the middle 1970s. This paper intends to provide a concise introduction to this aspect of the theory of phase transitions which clarifies the deep statistical significance of critical universality. © 2001 Elsevier Science B.V. All rights reserved.

PACS: 02.50.-r; 05.10.Cc; 05.20.-y

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PII: S0370-1573(01)00042-4

1. Introduction

The renormalization group (RG) is both a way of thinking and a calculational tool which acquired its full maturity in connection with the theory of the critical point in phase transitions. The basic physical idea of the RG is that when we deal with systems with infinitely many degrees of freedom, like thermodynamic systems, there are relatively simple relationships between properties at different space scales so that in many cases we are able to write down explicit exact or approximate equations which allow us to study asymptotic behaviour at very large scales.

The first systematic use of probabilistic methods in statistical mechanics was made by Khinchin who showed, using the central limit theorem, that the Boltzmann distribution of the single-molecule energy in systems of weakly correlated molecules is universal, that is independent of the form of the interaction, provided it is of short range. In his well known book [1] he emphasizes that physicists had not fully appreciated the generality of probabilistic methods so that, for example, they provided a new derivation (usually heuristic) of the Boltzmann law for every type of interaction. Similar remarks apply to the first applications of the RG in statistical mechanics. RG was introduced as a tool to explain theoretically the universality phenomena, the scaling laws, discovered experimentally near the critical point of a phase transition. In the first period RG calculations used different formal devices, mostly borrowed from quantum field theory, which gave good qualitative and quantitative results. It was soon realized that a new class of limit theorems in probability was involved. This class referred to situations of strongly correlated variables to which the central limit theorem does not apply, that is situations opposite to those considered by Khinchin. In fact, it was discovered that a critical point can be characterized by deviations from the central limit theorem.

Before providing a description of the content of the present paper we give a short account of the development of RG ideas in statistical mechanics.

It is useful to distinguish two different conceptual approaches. The first use of the RG in the study of critical phenomena [2] was based on a Green's function approach to statistical mechanics which paralleled quantum field theory. We recall Eq. (1) of [2]

$$\mathcal{G}(x, \{y_i\}, \alpha) = Z(t, \{y_i\}, \alpha) \mathcal{G}(x/t, \{y_i/t\}, \alpha Z_V^{-1}(t, \{y_i\}, \alpha) Z^2(t, \{y_i\}, \alpha)) , \quad (1.1)$$

where \mathcal{G} is a dimensionless two-point Green's function depending on a momentum variable x , a set of physical parameters y_i and the intensity of the interaction α (all dimensionless). This is an exact generalized scaling relation which in the vicinity of the critical point reduces to the phenomenological scaling due to the disappearance of the irrelevant parameters. The scaling functions Z and Z_V can be expressed in terms of Green's functions themselves via certain normalization conditions. This equation provided a qualitative explanation of scaling and, after the introduction of a non-integer space dimension d and the use of $\epsilon = 4 - d$ as a perturbation parameter [3], became the basis for systematic quantitative calculations [4–6].

The second approach started with the use on the part of Wilson of a different notion of RG [7] that he had already introduced in a different context, the fixed source meson theory [8,9], with no reference to critical phenomena. This was akin to certain intuitive ideas of Kadanoff [10] about the mechanism of reduction of relevant degrees of freedom near the critical point. Kadanoff's idea was that in the critical regime a thermodynamic system, due to the strong

correlations among the microscopic variables, behaves as if constituted by blocks of arbitrary size. In Wilson's approach, in fact, the calculation of a statistical sum consisted in a progressive elimination of the microscopic degrees of freedom to obtain the asymptotic large scale properties of the system.

Formally, the Green's function and the Wilson method were very different and in particular the first one implied a true group structure while the second was a semigroup. Both gave exactly the same results and the problem arose of clarifying the conceptual structures underlying these methods. In fact, many people were confused by this situation and some thought that the two methods had little connection with each other. Actually, the possibility of different RG transformations equally effective in the study of critical properties could be easily understood using concepts from the theory of dynamical systems. The critical point corresponds to a fixed point of these transformations and the quantities of physical interest, i.e. the critical indices, are connected with the hyperbolic behaviour in its neighbourhood, which is preserved if the different transformations are related by a differentiable map [11]. Still the multiplicative structure of the Green's function RG and the elimination of degrees of freedom typical of Wilson's approach did not appear easy to reconcile. The formal connection was clarified in [12] where it was shown that to any type of RG transformation one can associate a multiplicative structure, a cocycle, and the characterizing feature of the Green's function RG is that it is defined directly in terms of this structure. In the probabilistic setting the multiplicative structure is related to the properties of conditional expectations as discussed in [30] and illustrated in the present paper. The relationship between the two approaches is an aspect that has not been fully appreciated in the literature and even an authoritative recent exposition of the RG history seems to suggest the existence of basic conceptual differences [13]. In particular, Fisher discusses whether equations like (1.1) did anticipate Wilson and concludes negatively. Different interpretations in the history of scientific ideas are, of course, legitimate but after 30 years of applications of RG in critical phenomena and other fields this conclusion does not appear justified. In his 1970 paper [9] on meson theories Wilson had emphasized analogies of his approach with the Green's function RG of Gell-Mann and Low even though a detailed comparison was not yet available. A balanced presentation of the different RG approaches to critical phenomena can be found in the second edition of [14] of which, to my knowledge, there is no English translation.

More difficult was to understand at a deeper level the statistical nature of the critical universality. After 25 years I still think that the language of probability provides the clearest description of what is involved. The first hint came from a study by Bleher and Sinai [15] of Dyson's hierarchical models where they showed that at the critical point the increase of fluctuations required a normalization different from the square root of the number of variables in order to obtain a non-singular distribution for sums of spin variables (block spin). This normalization factor is directly related to the rescaling parameter of the fields in the RG. The limit distribution could be either a Gaussian as in the central limit theorem (CLT) or a different one which could be calculated approximately. The next step consisted in the recognition that new limit theorems for random fields were involved [16–18].

The random fields appearing in these limit theorems have scaling properties and some examples had already appeared in the probabilistic literature. However these examples [19] were not of a kind natural in statistical mechanics. The new challenging problem posed by the theory of phase transitions was the case of short range interactions producing at the critical point long

range correlations whose scaling behaviour cannot be easily guessed from the microscopic parameters. A general theory of such limit theorems is still missing and so far rigorous progress has been obtained in situations which are not hierarchical but share with these the fact that some form of scaling is introduced from the beginning.

The main part of the present article will review the connections of RG with limit theorems as they were understood in the decade 1975–1985. The justification for presenting old material resides in the fact that these results are scattered in many different publications very often with different perspectives. Here an effort is made to present the probabilistic point of view in a synthetic and coherent way. Section 10 will give an idea of more recent work trying to extract some general feature from the hard technicalities which characterize it.

2. A renormalization group derivation of the central limit theorem

CLT asserts the following. Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of independent identically distributed (i.i.d.) random variables with finite variance $\sigma^2 = \mathbb{E}(\xi_i - \mathbb{E}(\xi_i))^2$, where \mathbb{E} means expectation with respect to their common distribution. Then

$$\frac{\sum_1^n (\xi_i - \mathbb{E}(\xi_i))}{\sigma n^{1/2}} \xrightarrow{n \rightarrow \infty} N(0, 1), \quad (2.1)$$

where the convergence is in law and $N(0, 1)$ is the normal centered distribution of variance 1. To visualize things consider the random variables ξ_i as discrete or continuous spins associated to the points of a one dimensional lattice \mathbf{Z} and introduce the block variables $\zeta_n^1 = 2^{-n/2} \sum_1^{2^n} \xi_i$ and $\zeta_n^2 = 2^{-n/2} \sum_{2^n+1}^{2^{n+1}} \xi_i$. Then

$$\zeta_{n+1} = \frac{1}{\sqrt{2}} (\zeta_n^1 + \zeta_n^2). \quad (2.2)$$

Therefore, we can write the recursive relation for the corresponding distributions

$$p_{n+1}(x) = \sqrt{2} \int dy p_n(\sqrt{2}x - y) p_n(y) = (\mathcal{R}p_n)(x). \quad (2.3)$$

The non-linear transformation \mathcal{R} is what we call a renormalization transformation. Let us find its fixed points, i.e. the solutions of the equation $\mathcal{R}p = p$. An easy calculation shows that the family of Gaussians

$$p_{G,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \quad (2.4)$$

are fixed points. To prove the CLT we have to discuss the conditions under which the iteration of \mathcal{R} converges to a fixed point of variance σ^2 . The standard analytical way is to use the Fourier transform since \mathcal{R} is a convolution. In view of the subsequent developments here we shall illustrate the mechanism of convergence in the neighbourhood of a fixed point from the point of view of non-linear analysis. There are three conservation laws associated with

\mathcal{R} : normalization, centreing and variance. In formulas

$$\int p_{n+1}(x) dx = \int p_n(x) dx , \quad (2.5)$$

$$\int x p_{n+1}(x) dx = \int x p_n(x) dx , \quad (2.6)$$

$$\int x^2 p_{n+1}(x) dx = \int x^2 p_n(x) dx . \quad (2.7)$$

Therefore, only distributions with variance σ^2 can converge to a Gaussian $p_{G,\sigma}(x)$. We fix $\sigma=1$ and write p_G for $p_{G,1}$. Let us write the initial distribution as a centred deformation of the Gaussian with the same variance

$$p_\eta(x) = p_G(x)(1 + \eta h(x)) , \quad (2.8)$$

where η is a parameter. The function $h(x)$ must satisfy

$$\int p_G(x) h(x) dx = 0 , \quad (2.9)$$

$$\int p_G(x) x h(x) dx = 0 , \quad (2.10)$$

$$\int p_G(x) x^2 h(x) dx = 0 . \quad (2.11)$$

Suppose now η small. In linear approximation we have

$$(\mathcal{R}p_\eta) = p_G(1 + \eta(\mathcal{L}h)) + \mathcal{O}(\eta^2) , \quad (2.12)$$

where \mathcal{L} is the linear operator

$$(\mathcal{L}h)(x) = 2\pi^{-1/2} \int dy e^{-y^2} h(y + x2^{-1/2}) . \quad (2.13)$$

The eigenvalues of \mathcal{L} are

$$\lambda_k = 2^{1-k/2} \quad (2.14)$$

and the eigenfunctions the Hermite polynomials. The three conditions above on $h(x)$ can be read as the vanishing of its projections on the first three Hermite polynomials.

The mechanism of convergence of the deformed distribution to the normal law is now clear in linear approximation: if we develop h in Hermite polynomials only terms with $k > 2$ will appear so that upon iteration of the RG transformation they will contract to zero exponentially as the corresponding eigenvalues are < 1 .

To complete the proof one must show that the non-linear terms do not alter the conclusion. This is less elementary and will not be pursued here.

A terminological remark. The Gaussian is an example of what is called in probability theory a stable distribution. These are distributions which are fixed points of convolution equations and, with the exception of the Gaussian, have infinite variance.

3. Hierarchical models

Suppose now that the ξ_i are not independent. A case which has played a very important role in the development of the RG theory of critical phenomena is that of hierarchical models. To keep the notation close to that of the previous section we write the recursion relation connecting the distribution at level n to that at level $n + 1$

$$p_{n+1}(x) = (\hat{\mathcal{R}} p_n)(x) = g_n(x^2)(\mathcal{R} p_n)(x), \quad (3.1)$$

where $g_n(x^2)$ is a sequence of positive increasing functions and \mathcal{R} has the same meaning as in the previous section. It is clear that such a dependence tends to favour large values of the block variable x and therefore values of the ξ_i 's of the same sign. We call this a ferromagnetic dependence. We make the following choice $g_n(x^2) = L_n e^{\beta(c/2)^n x^2}$, where the constant L_n is determined by the normalization condition. This type of recursion arises from the following Gibbs distribution:

$$d\mu = Z_n^{-1} e^{-\beta H_n(x_1, \dots, x_{2^n})} \prod_{i=1}^{2^n} dp_0(x_i), \quad (3.2)$$

where H_n has the following hierarchical structure:

$$H_n(x_1, \dots, x_{2^n}) = H_{n-1}(x_1, \dots, x_{2^{n-1}}) + H_{n-1}(x_{2^{n-1}+1}, \dots, x_{2^n}) - c^n \left(\sum_{i=1}^{2^n} \frac{x_i}{2} \right)^2, \quad (3.3)$$

$H_0 = 0$ and $p_0(x)$ is a single spin distribution which characterizes the model. The constant c satisfies $1 < c < 2$. For $c < 1$ the model is trivial while for $c > 2$ becomes thermodynamically unstable.

To understand what happens in the case of dependent variables let us consider the hierarchical model defined by a Gaussian single spin distribution where the iteration can be performed exactly, that is

$$p_0(x) = (2\pi)^{-1/2} e^{-x^2/2}, \quad (3.4)$$

$$(\hat{\mathcal{R}}^n p_0)(x) = (2\pi\sigma_n^2)^{-1/2} e^{-x^2/2\sigma_n^2}, \quad (3.5)$$

$$\sigma_n^2 = \left(1 - 2\beta \sum_{i=1}^n (c/2)^i \right)^{-1}. \quad (3.6)$$

We see that the conservation of variance does not hold anymore under the transformation $\hat{\mathcal{R}}$ and, in fact, the variance increases at each iteration and is β dependent. When n tends to infinity the iteration converges to the distribution

$$p(x) = (2\pi)^{-1/2} (1 - 2\beta c/(2 - c))^{1/2} e^{-(1 - 2\beta c/(2 - c))x^2/2} \quad (3.7)$$

provided $\beta < \beta_{\text{cr}} = 1/c - \frac{1}{2}$, that is the CLT holds if the temperature is sufficiently large. At $\beta = \beta_{\text{cr}}$ the variance of the limit distribution explodes which means that the fluctuations

increase faster than $\mathcal{O}(2^{n/2})$. We try a new normalization of the block variable and consider $\sum_1^{2^n} \xi_i / 2^n c^{-n/2}$. The recursion for the distribution of this variable is

$$p_{n+1}(x) = L_n e^{\beta x^2} \int dy p_n(2/c^{1/2}x - y) p_n(y). \quad (3.8)$$

We now follow the same pattern as in the previous section: calculate the fixed points and see whether they admit a stable manifold or, in probabilistic language, a domain of attraction. The fixed points are the solutions of the equation

$$p(x) = L e^{\beta x^2} \int dy p(2/c^{1/2}x - y) p(y) \quad (3.9)$$

with L determined by normalization. A Gaussian solution is easily found

$$p_G(x) = (a_0/\pi)^{1/2} e^{-a_0 x^2} \quad (3.10)$$

with $a_0 = c\beta/(2 - c)$. We shall again discuss stability in linear approximation by considering a centred deformation of (3.10) $p_\eta(x) = p_G(x)(1 + \eta h(x))$. For small η

$$(\hat{\mathcal{H}} p_\eta) = p_G(1 + \eta(\hat{\mathcal{L}} h)) + \mathcal{O}(\eta^2), \quad (3.11)$$

where $\hat{\mathcal{L}}$ is the linear operator

$$(\hat{\mathcal{L}} h)(x) = \int dy e^{-2a_0 y^2} (h((x + y)/c^{1/2}) + h((x - y)/c^{1/2})). \quad (3.12)$$

The corresponding eigenvalues for even eigenfunctions are $\lambda_{2k} = 2/c^k$, with $k = 0, 1, 2, 3, \dots$, and the eigenfunctions are rescaled Hermite polynomials of even degree.

Here $h(x)$ must be considered as the effect of many iterations starting from some initial distribution which characterizes the model and is therefore dependent on β . We now see that for $2 > c > 2^{1/2}$ the eigenvalues λ_0 and λ_2 are > 1 . The projection of h over the constants vanishes due to the normalization condition so that for the iteration to converge we have to impose the vanishing of the projection of h on the second Hermite polynomial. In view of the previous remark this will select a special value β_{cr} , the critical temperature for the model considered. In conclusion, for $2 > c > 2^{1/2}$ and $\beta = \beta_{\text{cr}}$ the fixed point (3.10) has a non-empty domain of attraction.

When $c < 2^{1/2}$ the Gaussian fixed point becomes unstable and we must investigate about the existence of other fixed points. Bifurcation theory tells us that most likely there is an exchange of stability between two fixed points and we should look for the new one in the direction which has just become unstable. For $c < 2^{1/2}$ we have $\lambda_4 > 1$ so the instability is in the direction H_4 , the Hermite polynomial of fourth order. Define $\epsilon = 2^{1/2} - c$ and look, for ϵ small, for a solution of (3.9) of the form

$$p_{\text{NG}}(x) = p_G(x)(1 - \epsilon a H_4(\gamma x)) + \mathcal{O}(\epsilon^2) \approx e^{-r^*(\epsilon)x^2/2 - u^*(\epsilon)x^4/4}, \quad (3.13)$$

where r^* and u^* are the fixed point couplings. The analysis of this case is considerably more complicated and gives the following results: the linearization of the RG transformation around (3.13) has only one unstable direction so that by requiring the vanishing of an appropriate projection along this direction we obtain a non empty domain of attraction for some β_{cr} [20,21].

4. Eigenvalues of the linearized RG and critical indices

We illustrate the interpretation of the eigenvalues of the linearized RG at a fixed point in the context of hierarchical models, which is especially simple. Notations are as in the previous section. Consider for definiteness the region $2^{1/3} < c < 2^{1/2}$ so that the Gaussian fixed point is unstable, but in its neighbourhood there exists a non-trivial non-Gaussian fixed point of the form (3.13) with a non-empty domain of attraction. Suppose now that we start the iteration of the RG from some initial distribution which is close to it but not in its domain of attraction. For example we may consider a distribution $p(x, \beta)$ of the form (3.13) with parameters r, u slightly different from r^*, u^* which are the values taken at the critical temperature β_{cr} . Application of the RG transformation will eventually drive this distribution away from the fixed point due to the presence of an unstable direction. However, we can “renormalize” our parameters r, u by compensating the instability at each iteration and find a sequence of parameters r_n, u_n such that when n tends to infinity we have a sequence of distributions $p_n(x, \beta_n)$ approaching a definite limit. Since we have assumed that the parameters r_n, u_n are close to the fixed point values, the renormalized parameters can be simply expressed in terms of rescalings determined by the eigenvalues of a linear operator analogous to $\hat{\mathcal{L}}$ introduced in the previous section in connection with the Gaussian fixed point. This can be seen as follows. Let us write our initial distribution as a deformation of (3.13)

$$p(x, \beta) = p_{\text{NG}}(x, \beta_{\text{cr}})(1 + \eta h(x, \beta)) . \quad (4.1)$$

If we develop h in terms of eigenfunctions of the linearized RG at (3.13), the iteration of the RG transformation will multiply the projections of h along these eigenfunctions by powers of the corresponding eigenvalues. The explosion in the unstable direction can then be controlled by rescaling at each step the projection in this direction with a factor proportional to the inverse eigenvalue.

Let us define the susceptibility of a block of 2^n spins

$$\chi_n(\beta) = \frac{1}{2^n} \mathbb{E} \left[\left(\sum_1^{2^n} \xi_i \right)^2 \right] . \quad (4.2)$$

This quantity can be easily expressed in terms of the distribution $p_n(x, \beta)$ of the block with the critical normalization $2^n c^{-n/2}$

$$\chi_n(\beta) = (2/c)^n \int dx x^2 p_n(x, \beta) . \quad (4.3)$$

As $n \rightarrow \infty$, χ_n diverges if $\beta = \beta_{\text{cr}}$.

To calculate the susceptibility critical index as β approaches the critical value we assume that the two limits $n \rightarrow \infty$ and $\beta \rightarrow \beta_{\text{cr}}$ can be interchanged so that they can be calculated over subsequences. We want to compute

$$v_\chi = \lim_{n \rightarrow \infty} \frac{\log \chi_n(\beta_n)}{\log |\beta_n - \beta_{\text{cr}}|} = \lim_{n \rightarrow \infty} \frac{-n \log(c/2) + \log \int dy y^2 p_n(y, \beta_n)}{\log |\beta_n - \beta_{\text{cr}}|} . \quad (4.4)$$

If we now choose $|\beta_n - \beta_{\text{cr}}| \approx \lambda^{-n}$, where λ is the eigenvalue corresponding to the unstable direction of the RG linearized at the fixed point, the integral appearing in this formula will be almost constant and we obtain

$$v_\chi = \frac{\log(c/2)}{\log \lambda} . \quad (4.5)$$

Similar calculations can be done for other thermodynamical quantities like the free energy or the magnetization.

We can summarize the situation as follows: given a model defined by an initial distribution p_0 , for $\beta < \beta_{\text{cr}}$ we expect the CLT to hold. For $\beta = \beta_{\text{cr}}$ by properly normalizing the block variables we have new limit theorems where the limit law has a domain of attraction which is a non-trivial submanifold in the space of probability distributions called the critical manifold. If we start from a distribution which is not in the domain of attraction of a given fixed point, but not too far from it, it can still be driven to a regular limit by rescaling at each step its coefficients in a way dictated by the fixed point. This defines the so-called scaling limits of the theory associated to a given fixed point.

For further reading see [21–23].

5. Self-similar random fields

The notion of self similar random field was introduced informally in [16] and rigorously in [17] and independently in [18]. It was then developed more systematically in [24,25]. The idea was to construct a proper mathematical setting for the notion of RG à la Kadanoff–Wilson. This led to a generalization of limit theorems for random fields to the situation in which the variables are strongly correlated.

Let \mathbf{Z}^d be a lattice in d -dimensional space and j a generic point of \mathbf{Z}^d , $j = (j_1, j_2, \dots, j_d)$ with integer coordinates j_i . We associate to each site a centred random variable ξ_j and define a new random field

$$\xi_j^n = (\mathcal{R}_{\alpha,n} \xi)_j = n^{-d\alpha/2} \sum_{s \in V_j^n} \xi_s , \quad (5.1)$$

where

$$V_j^n = \{s : j_k n - n/2 < s_k \leq j_k n + n/2\} \quad (5.2)$$

and $1 \leq \alpha < 2$. Transformation (5.1) induces a transformation on probability measures according to

$$(\mathcal{R}_{\alpha,n}^* \mu)(A) = \mu'(A) = \mu(\mathcal{R}_{\alpha,n}^{-1} A) , \quad (5.3)$$

where A is a measurable set and $\mathcal{R}_{\alpha,n}^*$ has the semigroup property

$$\mathcal{R}_{\alpha,n_1}^* \mathcal{R}_{\alpha,n_2}^* = \mathcal{R}_{\alpha,n_1+n_2}^* . \quad (5.4)$$

A measure μ will be called self-similar if

$$\mathcal{R}_{\alpha,n}^* \mu = \mu \quad (5.5)$$

and the corresponding field will be called a self-similar random field. We briefly discuss the choice of the parameter α . It is natural to take $1 \leq \alpha < 2$. In fact, $\alpha = 2$ corresponds to the law of large numbers so that the block variable (5.1) will tend for large n to zero in probability. The case $\alpha > 1$ means that we are considering random systems which fluctuate more than a collection of independent variables and $\alpha = 1$ corresponds to the CLT. Mathematically, the lower bound is not natural but it becomes so when we restrict ourselves to the consideration of ferromagnetic-like systems.

A general theory of self-similar random fields so far does not exist and presumably is very difficult. However, Gaussian fields are completely specified by their correlation function and self-similar Gaussian fields can be constructed explicitly [18,26]. It is easier if we represent the correlation function in terms of its Fourier transform

$$\mathbb{E}(\xi_i \xi_j) = \int_{-\pi}^{\pi} \prod_1^d d\lambda_k \rho(\lambda_1, \dots, \lambda_d) e^{i \sum_k \lambda_k (i-j)_k} . \quad (5.6)$$

The prescription to construct ρ in such a way that the corresponding Gaussian field satisfies (5.5) is as follows. Take a positive homogeneous function $f(\lambda_1, \dots, \lambda_d)$ with homogeneity exponent $d(1 + \alpha)$, that is

$$f(c\lambda_1, \dots, c\lambda_d) = c^{d(1+\alpha)} f(\lambda_1, \dots, \lambda_d) . \quad (5.7)$$

Next, we construct a periodic function $g(\lambda_1, \dots, \lambda_d)$ by taking an average over the lattice Z^d

$$g(\lambda_1, \dots, \lambda_d) = \sum_{i_k} \frac{1}{f(\lambda_1 + i_1, \dots, \lambda_d + i_d)} . \quad (5.8)$$

If we take now

$$\rho(\lambda_1, \dots, \lambda_d) = \prod_i |1 - e^{i\lambda_i}|^2 g(\lambda_1, \dots, \lambda_d) , \quad (5.9)$$

it is not difficult to see that the corresponding Gaussian measure satisfies (5.5). The periodicity of ρ insures translational invariance.

For $d = 1$ there is only one, apart from a multiplicative constant, homogeneous function and one can show that the above construction exhausts all possible Gaussian self-similar distributions. For $d > 1$ it is not known whether a similar conclusion holds.

From this point on one can follow in the discussion the same pattern as for hierarchical models and investigate the stability of the Gaussian fixed points P_G of (5.5). Consider a deformation $P_G(1 + h)$ and the action of $\mathcal{R}_{\alpha,n}^*$ on this distribution. It is easily seen that

$$\mathcal{R}_{\alpha,n}^* P_G h = \mathbb{E}(h | \{\xi_j^n\}) \mathcal{R}_{\alpha,n}^* P_G = \mathbb{E}(h | \{\xi_j^n\}) P_G(\{\xi_j^n\}) . \quad (5.10)$$

The conditional expectation on the right hand side of (5.10) will be called the linearization of the RG at the fixed point P_G . To proceed further in the study of the stability we have to find its eigenvectors and eigenvalues. These have been calculated by Sinai. The eigenvectors are appropriate infinite dimensional generalizations of Hermite polynomials H_k which are described in full detail in [26]. They satisfy the eigenvalue equation

$$\mathbb{E}(H_k | \{\xi_j^n\}) = n^{[k(\alpha/2-1)+1]d} H_k(\{\xi_j^n\}) . \quad (5.11)$$

We see immediately that H_2 is always unstable while H_4 becomes unstable when α crosses from below the value $\frac{3}{2}$. By introducing the parameter $\epsilon = \alpha - \frac{3}{2}$, in principle, one can construct, as in the hierarchical case, a non-Gaussian fixed point. The formal construction is explained in Sinai's book [26] where one can find also an exhaustive discussion of the questions, mostly unsolved, arising in this connection. A different construction of a non-Gaussian fixed point, for $d=4$ has been made recently by Brydges, Dimock and Hurd. This will be briefly discussed in Section 10.

6. Some properties of self-similar random fields

We have already characterized the critical point as a situation of strongly dependent random variables, in which the CLT fails. We want to give here a characterization which refers to the random field globally. Consider in the product space of the variables ξ_i the cylinder sets, that is the sets of the form

$$\{\xi_{i_1} \in A_1, \dots, \xi_{i_n} \in A_n\}, \quad (6.1)$$

with $i_1, \dots, i_n \in A$, A being an arbitrary finite region in \mathbb{Z}^d and the A_i measurable sets in the space of the variables ξ_i . We denote with Σ_A the σ -algebra generated by such sets. We say that the variables ξ_i are *weakly dependent* or that they are a *strong mixing* random field if the following holds. Given two finite regions A_1 and A_2 separated by a distance

$$d(A_1, A_2) = \min_{i \in A_1, j \in A_2} |i - j|, \quad (6.2)$$

where $|i - j|$ is for example the Euclidean distance, define

$$\tau(A_1, A_2) = \sup_{A \in \Sigma_{A_1}, B \in \Sigma_{A_2}} |\mu(A \cap B) - \mu(A)\mu(B)|. \quad (6.3)$$

Then $\tau(A_1, A_2) \rightarrow 0$ when $d(A_1, A_2) \rightarrow \infty$.

Intuitively, the strong mixing idea is that one cannot compensate for the weakening of the dependence of the variables due to an increase of their space distance, by increasing the size of the sets.

This situation is typical when one has exponential decay of correlations. This has been proved for a wide class of random fields including ferromagnetic non-critical spin systems [27].

The situation is entirely different at the critical point where one expects the correlations to decay as an inverse power of the distance. In this connection the following result has been proved in [28]: a ferromagnetic translational invariant system with pair interactions with correlation function

$$C(i) = \mathbb{E}(\xi_0 \xi_i) - \mathbb{E}(\xi_0) \mathbb{E}(\xi_i) \quad (6.4)$$

such that

$$\lim_{L \rightarrow \infty} \frac{\sum_{L(s_k-1) \leq i_k < L(s_k+1)} C(i)}{\sum_{0 \leq i_k < L} C(i)} \neq 0 \quad (6.5)$$

for arbitrary s_k , does not satisfy the strong mixing condition.

This theorem implies in particular that a critical two-dimensional Ising model violates strong mixing. Therefore, violation of strong mixing seems to provide a reasonable characterization of the type of strong dependence encountered in critical phenomena. On the other end, under very general conditions, if strong mixing holds the one-block distribution satisfies the CLT [29].

An interesting question is whether we can describe the structure of the limit one-block distributions that can appear at the critical point beside the Gaussian. It was shown in [28], building on previous results by Newman, that for ferromagnetic systems the Fourier transform (characteristic function in probabilistic language) of the limit distribution must be of the form

$$\mathbb{E}(e^{it\xi}) = e^{-bt^2} \prod_j (1 - t^2/\alpha_j^2) \quad (6.6)$$

with $\sum_j 1/\alpha_j^2 < \infty$. In the probabilistic literature these distributions are called the *D*-class [36]. The Gaussian is the only infinitely divisible distribution belonging to this class.

7. Multiplicative structure

In this section we show that there is a natural multiplicative structure associated with transformations on probability distributions like those induced by the RG. This multiplicative structure is related to the properties of conditional expectations. We use the notations of Section 5. Suppose we wish to evaluate the conditional expectation

$$\mathbb{E}(h|\{\xi_j^n\}), \quad (7.1)$$

where the collection of block variables ξ_j^n indexed by j is given a fixed value. Here h is a function of the individual spins ξ_i . It is an elementary property of conditional expectations that

$$\mathbb{E}(\mathbb{E}(h|\{\xi_j^n\})|\{\xi_j^{nm}\}) = \mathbb{E}(h|\{\xi_j^{nm}\}). \quad (7.2)$$

Let P be the probability distribution of the ξ_i and $\mathcal{R}_{\alpha,n}^*P$ the distribution obtained by applying the RG transformation, that is the distribution of the block variables ξ_j^n . By specifying in (7.2) the distribution with respect to which expectations are taken we can rewrite it as

$$\mathbb{E}_{\mathcal{R}_{\alpha,n}^*P}(\mathbb{E}_P(h|\{\xi_j^n\})|\{\xi_j^{nm}\}) = \mathbb{E}_P(h|\{\xi_j^{nm}\}). \quad (7.3)$$

This is the basic equation of this section and we want to work out its consequences. For this purpose we generalize the eigenvalue equation (5.11) to the case in which the probability distribution is not a fixed point of the RG. In analogy with the theory of dynamical systems we interpret the conditional expectation as a linear transformation from the linear space tangent to P to the linear space tangent to $\mathcal{R}_{\alpha,n}^*P$ and we assume that in each of these spaces there is a basis of vectors $H_k^P, H_k^{\mathcal{R}_{\alpha,n}^*P}$ connected by the following generalized eigenvalue equation [31]:

$$\mathbb{E}_P(H_k^P|\{\xi_j^n\}) = \lambda_k(n, P) H_k^{\mathcal{R}_{\alpha,n}^*P}(\{\xi_j^n\}). \quad (7.4)$$

Eq. (7.3) implies that the λ 's must satisfy the relationship

$$\lambda_k(m, \mathcal{R}_{\alpha,n}^* P) \lambda_k(n, P) = \lambda_k(mn, P). \quad (7.5)$$

From (7.4) and (7.5) we find that the λ_k are given by the following expectations:

$$\lambda_k(n, P) = \mathbb{E}(\bar{H}_k^{\mathcal{R}_{\alpha,n}^* P}(\{\xi_j^n\}) H_k^P(\{\xi_j\})), \quad (7.6)$$

where \bar{H}_k^P are dual to H_k^P according to the orthogonality relation $\int \bar{H}_k^P H_j^P dP = \delta_{kj}$. The λ_k are therefore special correlation functions. The similarity between Eqs. (7.5) and (1.1) is then obvious. The Green's function RG corresponds to a very simple transformation on the probability distribution such that its form is unchanged and only the values of its parameters are modified.

8. RG and effective potentials

In this section we want to illustrate a connection between RG and the theory of large deviations [32]. By large deviations we mean fluctuations with respect to the law of large numbers, e.g., fluctuations of the magnetization in a large but finite volume. In view of the connection of RG with limit theorems our discussion will parallel, actually generalize, some well known facts in the theory of sums of independent random variables. This will lead to a probabilistic interpretation of a widely used concept in physics, the effective potential, and will clarify its relationship with the effective Hamiltonian in RG theory. We continue with our model system of continuous spins ξ_i indexed by the sites of a lattice in d dimensions and try to estimate the probability that the magnetization in a volume A be larger than zero at some temperature above criticality. From the exponential Chebyshev inequality we have for $\theta > 0, x > 0$

$$P\left(\sum_{i \in A} \xi_i / |A| \geq x\right) \leq e^{-|A|\theta x} \mathbb{E}(e^{\theta \sum_{i \in A} \xi_i}) \leq e^{-|A|\Gamma(|A|, x)}, \quad (8.1)$$

where

$$\Gamma(|A|, x) = \sup_{\theta > 0} \left(\theta x - \frac{1}{|A|} \log \mathbb{E}(e^{\theta \sum_{i \in A} \xi_i}) \right) \quad (8.2)$$

is the Legendre transform of $(1/|A|) \log \mathbb{E}(e^{\theta \sum_{i \in A} \xi_i})$. With some more work one can establish also a lower bound

$$P\left(\sum_{i \in A} \xi_i / |A| \geq x\right) \geq e^{-|A|(\Gamma(|A|, x) + \alpha(|A|) + \delta)} \quad (8.3)$$

with $\alpha \rightarrow 0$ for $A \rightarrow \infty$, and $\delta > 0$ arbitrarily small. We then conclude

$$-\lim_{A \rightarrow \infty} \frac{1}{|A|} \log P\left(\sum_{i \in A} \xi_i / |A| \geq x\right) = \lim_{A \rightarrow \infty} \Gamma(|A|, x) = V_{\text{eff}}(x), \quad (8.4)$$

where $V_{\text{eff}}(x)$ is known in the physical literature as the effective potential. An important remark. While $\Gamma(|A|, x)$, being the Legendre transform of a convex function, is always convex for any

A , this is not the case with $-(1/|A|)\log P(\sum_{i \in A} \xi_i/|A| \geq x) = V(|A|, x)$ for finite A and one has to be careful in interpreting, for example, results from numerical simulations.

To understand the connection with the RG it is convenient to consider first the case of independent random variables, that is the situation considered in Section 2. A classical problem in limit theorems for independent variables is the estimate of the corrections to the CLT when the argument of the limit distribution increases with n . A well known result in this domain is the following [29]: suppose we want to estimate $P(\sum_1^n \xi_i/n^{1/2} \geq x)$ when $x = o(n^{1/2})$. Then

$$P\left(\sum_1^n \xi_i/n^{1/2} \geq x\right) \approx e^{-n \sum_2^s \Gamma_k(xn^{-1/2})^k} \quad (8.5)$$

for $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} x^{s+1}n^{-(s-1)/2} = 0$. The function $\Gamma(z) = \sum_2^\infty \Gamma_k z^k$ is the Legendre transform of $\log \mathbb{E}(e^{\theta \xi_i})$. The sign \approx has to be understood as logarithmic dominance. If $x = \mathcal{O}(n^{1/2})$, the whole function Γ contributes and we are back to the large deviation estimate at the beginning of this section.

We expect a result like (8.5) to hold for the one-block distribution in the case of dependent variables as in statistical mechanics away from the critical point. We then see that the coefficients of an expansion of $\Gamma(|A|, x)$ in powers of x determine the corrections to the CLT for the one-block distribution. More interesting is the situation at the critical point. Suppose first that the one block limit distribution is Gaussian but the normalization is anomalous as it is the case in hierarchical models for a range of values of the parameter c . Instead of (8.5) we expect an estimate of the form

$$P\left(\sum_{i \in A} \xi_i/|A|^\rho \geq x\right) \approx e^{-|A| \sum_2^s \Gamma_k(|A|)(x/|A|^{1-\rho})^k} \quad (8.6)$$

with $\rho > \frac{1}{2}$ and $\lim_{|A| \rightarrow \infty} x^{s+1}/|A|^{(1-\rho)s-\rho} = 0$. We see that for the quadratic term to survive the coefficient Γ_2 must vanish when $|A| \rightarrow \infty$ as $|A|^{1-2\rho}$. If the one-block limit distribution is not Gaussian we can establish a general relationship between its logarithm and the effective potential. Let us write $-\log P = V_{\text{RG}}$ where P is the limit distribution. We now rewrite the large deviation estimate in the following way:

$$P\left(\sum_{i \in A} \xi_i/|A|^\rho \geq x|A|^{1-\rho}\right) \approx e^{-|A|\Gamma(|A|, x)} \quad (8.7)$$

Scale $x \rightarrow x/|A|^{1-\rho}$. Then we obtain

$$V_{\text{RG}}(x) = \lim_{|A| \rightarrow \infty} |A|\Gamma(|A|, x/|A|^{1-\rho}) \quad (8.8)$$

Therefore $\Gamma(|A|, x)$ determines in different limits either V_{eff} or V_{RG} . The discussion in the present section can be made rigorous in the case of hierarchical models.

9. Coexistence of phases in hierarchical models

In the case of hierarchical models the RG recursion relation for the one-block probability distribution can be easily rewritten as a recursion for the quantity $V(|A|, x)$ introduced in the

previous section which coincides with the effective potential in the limit $|A| \rightarrow \infty$. In fact if we normalize the block-spin with its volume, that is consider the mean magnetization, a simple calculation gives the following iteration for the corresponding probability distribution $\pi_n(x)$:

$$\pi_n(x) = L_n e^{\beta c^n x^2} \int dx' \pi_{n-1}(2x - x') \pi_{n-1}(x'). \quad (9.1)$$

Taking the logarithm and dividing by the number of spins 2^n , we obtain

$$V_n(x) = -\frac{1}{2} \log L_n - \beta \left(\frac{c}{2}\right)^n x^2 - \frac{1}{2} \log \int dx' e^{-2^{n-1}(V_{n-1}(2x-x') + V_{n-1}(x'))}. \quad (9.2)$$

To illustrate the difference between (9.2) and (3.8) let us consider again the simple case in which the model is defined by a Gaussian single spin distribution. The iteration of (9.2) gives for small β

$$V_n(x) = \frac{1}{2} \left[1 - \beta \sum_0^n (c/2)^k \right] x^2 + v_n, \quad (9.3)$$

where v_n tends to zero when $n \rightarrow \infty$. In this limit then

$$V_{\text{eff}} = \frac{1}{2} [1 - 2c\beta/(2 - c)] x^2. \quad (9.4)$$

The critical temperature is defined as the value β_{cr} for which the coefficient of x^2 vanishes and coincides with that found in Section 3. On the other hand, in that section, it was the only temperature for which recursion (3.8) converges to the Gaussian fixed point, i.e. the only temperature for which the following difference between two diverging expressions converges:

$$2\beta \sum_0^n \left(\frac{2}{c}\right)^k - \left(\frac{2}{c}\right)^{n+1}. \quad (9.5)$$

We want to apply now (9.2) to the study of the magnetization in the phase coexistence region for a general hierarchical model [33,34]. The problem we want to discuss is the following. In the hierarchical case at level n we have blocks, containing each 2^{n-1} spins, interacting in pairs through the Hamiltonian

$$c^n (\zeta_{n-1}^1 + \zeta_{n-1}^2)^2 / 4 = c^n (\zeta_n)^2, \quad (9.6)$$

where $\zeta_{n-1}^1, \zeta_{n-1}^2, \zeta_n$ are mean magnetizations. Suppose now that ζ_n is assigned the value αM , $0 < \alpha < 1$, M being the spontaneous magnetization corresponding to the temperature β . We want to calculate the conditional distribution of ζ_{n-1} given ζ_n , for large n . The remarkable result is that one of the quantities ζ_{n-1}^1 or ζ_{n-1}^2 with probability close to 1 is equal to the full magnetization M .

To compute the desired distribution we have to estimate $\pi_n(x)$ or, what is the same, $V_n(x)$ for large n . From (9.2) we expect asymptotically

$$V_n(x) = V_{\text{eff}}(x) + (c/2)^n Y(x) + \dots \quad (9.7)$$

Since in the phase coexistence region $V_{\text{eff}}(x)$ is flat, i.e. constant, the whole x dependence is given by $Y(x)$. In order to compute this function we perform a subtraction and consider

$V_n(x) - V_n(x_0)$ choosing x_0 in the flat region of $V_{\text{eff}}(x)$. From (9.2) it is easily seen that the quantity

$$\Delta_n(x) = (2/c)^n (V_n(x) - V_n(x_0)) \quad (9.8)$$

satisfies a recursion of the form

$$\Delta_n(x) = -c^{-n} \log A_n - \beta x^2 - c^{-n} \log \int dx' e^{-c^{n-1}(\Delta_{n-1}(x+x') + \Delta_{n-1}(x-x'))}, \quad (9.9)$$

where A_n is determined by the condition $\Delta_n(x_0) = 0$. Let us choose $x_0 = M$. By symmetry $\Delta_n(\pm M) = 0$. For $0 \leq x \leq M$ and large n the main contribution to the integral on the right hand side of (9.9) comes from the region $x \pm x' \approx M$, while for $-M \leq x \leq 0$ from the region $x \pm x' \approx -M$. We can write therefore the approximate recursion equations

$$\Delta_n(x) = \beta(M^2 - x^2) + c^{-1} \Delta_{n-1}(2x \mp M), \quad (9.10)$$

where the \mp in the second term on the right corresponds to $0 \leq x \leq M$ or $-M \leq x \leq 0$. This type of equations has been rigorously studied by Bleher [33] and the asymptotic solutions show a complicated fractal structure.

The conditional probability of interest to us is

$$P(|\zeta_{n-1} - M| < \epsilon M | \zeta_n = \alpha M) = \frac{\int_{|x'-M| < \epsilon M} dx' e^{-c^{n-1}(\Delta_{n-1}(x') + \Delta_{n-1}(2\alpha M - x'))}}{\int_{-\infty}^{\infty} dx' e^{-c^{n-1}(\Delta_{n-1}(x') + \Delta_{n-1}(2\alpha M - x'))}}. \quad (9.11)$$

Since the main contribution to the integral in the denominator comes from the same region appearing in the numerator, our conditional probability is for sufficiently large n as close as we want to 1.

10. Weak perturbations of Gaussian measures: a non-Gaussian fixed point

Starting at the end of the seventies the RG has become a very important and effective tool for proving rigorous results in statistical mechanics and Euclidean quantum field theory. An impressive amount of work has been done and it is not possible to give even a schematic account of it [35]. Many different versions of the RG idea have been used, each author or group of authors following his own linguistic propensities. Probability theory is always in the background and we want to try to recover some conceptual feature common to all of them. As in the previous part of this review limit theorems will be a relevant reference. However the limit theorems to be considered are of a different kind, they are those which in probability are called non-classical and are related to the following problem.

Given a probability distribution P and an integer n , can one consider it as resulting from the composition (convolution) of n distributions P_k , $k = 1, 2, \dots, n$? In other words, can one consider the random variable described by P as the sum of n independent random variables? In formulas

$$P = P_1 \star P_2 \star \dots \star P_n, \quad (10.1)$$

where \star means convolution. It is clear, for example, that a Gaussian distribution of variance σ^2 can be thought as the composition of any number n of Gaussians with variances σ_i^2 provided

$\sum_1^n \sigma_i^2 = \sigma^2$. The problem arises naturally of investigating under what conditions convolutions like the right hand side of (10.1) converge to a regular distribution as $n \rightarrow \infty$. The right hand side of (10.1) can be considered as a recurrence relation

$$\hat{P}_{n+1} = \hat{P}_n \star P_n, \quad (10.2)$$

where $\hat{P}_n = P_1 \star P_2 \star \cdots \star P_{n-1}$. Comparing (2.3) with (10.1) we see that while in the case of limit theorems for independent identically distributed random variables we have a natural fixed point problem, this is not in general the case for non-identically distributed variables. As we shall see below, the RG approach to Euclidean field theory and the statistical mechanics of the critical point has led to formulations which have analogies with these problems. In fact, infinite dimensional equations structurally similar to (10.2) are constructed which can be transformed into equations admitting fixed points after a rescaling.

In the following exposition, we shall follow the recent article by Brydges et al. [37]. The goal of these authors is the construction of a quantum field theory in \mathbf{R}^4 with non-trivial scaling behaviour at long distances, that is in the infrared region, determined by a non-Gaussian fixed point of an appropriate RG transformation. The starting point is a ϕ^4 theory in finite volume regularized at small distances to eliminate ultraviolet singularities. This model is believed to have a non-Gaussian fixed point in $4 - \epsilon$ dimensions and to simulate such a situation in 4 dimensions the authors introduce a special covariance for the Gaussian part of the measure. The first step consists in the construction of a covariance $v(x - y)$ which behaves at large distances like $(-A)^{-1-\epsilon/2}$. This means that it scales like $|x|^{-2+\epsilon}$ for large $|x|$. Their choice is

$$v(x - y) = \int_1^\infty d\alpha \alpha^{\epsilon/2-2} e^{-|x-y|^2/4\alpha}. \quad (10.3)$$

Such a covariance can be decomposed in the following way:

$$v(x - y) = \sum_{j=0}^\infty L^{-(2-\epsilon)j} C(L^{-j}(x - y)), \quad (10.4)$$

where $L > 1$ is a scaling factor and

$$C(x) = \int_1^{L^2} d\alpha \alpha^{\epsilon/2-2} e^{-|x|^2/4\alpha}. \quad (10.5)$$

Each term in the expansion can be interpreted as the covariance of a rescaled field

$$\phi_{L^{-j}}(x) = L^{-(2-\epsilon)j/2} \phi(L^{-j}x), \quad (10.6)$$

which has reduced fluctuations and varies over larger distances. The aim is to study the measure

$$d\mu_A = Z^{-1} e^{-V_A(\phi)} d\mu_v, \quad (10.7)$$

where

$$V_A(\phi) = \lambda \int_A : \phi^4 :_v + \zeta \int_A : (\partial\phi)^2 :_v + \mu \int_A : \phi^2 :_v \quad (10.8)$$

when A tends to \mathbf{R}^4 . The double dots indicate the Wick polynomials with respect to the covariance v .

Take for Λ a large cube of side L^N so that the measure is well defined. We want to calculate

$$(\mu_v \star e^{-V})(\phi) = (\mu_{\hat{C}_N} \star \mu_{\hat{C}_{N-1}} \star \cdots \star \mu_{\hat{C}_0} \star e^{-V})(\phi) \quad (10.9)$$

having used the above decomposition of the covariance with

$$\hat{C}_j(x) = L^{-(2-\epsilon)j} C(L^{-j}(x-y)) . \quad (10.10)$$

Actually, in finite volume we should specify some boundary conditions but we shall ignore this aspect. Next by defining

$$\hat{Z}_j(\phi) = (\mu_{\hat{C}_{j-1}} \star \cdots \star \mu_{\hat{C}_0} \star e^{-V})(\phi) , \quad (10.11)$$

we find the recursion relation

$$\hat{Z}_{j+1}(\phi) = (\mu_{\hat{C}_j} \star \hat{Z}_j)(\phi) . \quad (10.12)$$

In this way, the calculation is performed by successive integrations over variables which exhibit decreasing fluctuations. This is not yet our RG equation because as $j \rightarrow \infty$, $\mu_{\hat{C}_j}$ becomes a singular distribution and we do not obtain a fixed point equation. However by introducing the rescaled fields $\phi_{L^{-j}}(x) = L^{-(2-\epsilon)j/2} \phi(L^{-j}x)$ and the rescaled Z 's

$$Z_j(\phi) = \hat{Z}_j(\phi_{L^{-j}}) \quad (10.13)$$

the recursion becomes

$$Z_{j+1}(\phi) = (\mu_{\hat{C}} \star Z_j)(\phi_{L^{-1}}) \quad (10.14)$$

with initial condition $Z_0 = e^{-V}$. We emphasize that the last step is possible due to the special structure of the measures $\mu_{\hat{C}_j}$. It is now meaningful to look for the fixed points of (10.14). Brydges, Dimock and Hurd have proved that in $d=4$ for ϵ small there exists a non-Gaussian fixed point of (10.14) characterized by a value $\hat{\lambda}(\epsilon, L)$ of the coupling λ and that for certain values $\mu(\lambda), \zeta(\lambda)$ the iteration of (10.14) with initial condition e^{-V} converges to this fixed point. Technically, the proof is very complicated and its description is beyond the aims of this review. A very good exposition with some simplifications of the techniques employed can be found also in [38].

11. Concluding remarks

The question we want to consider is the following: which are the benefits for our understanding of critical phenomena and more generally of statistical physics deriving from the use of probabilistic language? Feynman thought that it is worth to spend one's time formulating a theory in every physical and mathematical way possible. In our case there is an intuition associated with probabilistic reasoning that is foreign to the usual formalisms of statistical mechanics based on correlation functions and equations connecting them.

Apart from this general remark we must consider that the rigorous results obtained so far in RG theory have been strongly influenced by the probabilistic language as this appears the most natural for the mathematical study of statistical mechanics and Euclidean field theory when a functional integral approach is used. New technical ideas however are needed to deal with

concrete problems like calculating the critical indices of the three-dimensional Ising model or establishing in a conclusive way whether the field theory ϕ_4^4 is ultraviolet non-trivial.

The formal apparatus of RG has been easily extended to the analysis of fermionic systems when these are described by a Grassmannian functional integral [39], that is by the analog of a Gibbs distribution over anticommuting variables. In this case the convergence of perturbation theory plays a major role on the way to rigorous results. Recently, it has been possible to give a true probabilistic expression to general Grassmannian integrals in terms of discrete jump processes (Poisson processes) [40,41], so that classical probability may become a main tool also in the study of fermionic systems especially in view of developing non-perturbative methods. For an early example of connection between anticommutative calculus and probability see [42].

In a wider perspective one may remark that the theory of Gibbs distributions is becoming instrumental also in various sectors of mathematical statistics, for example in image reconstruction, and critical situations appear also in this domain. Transfer of ideas from statistical mechanics to stochastic analysis is currently an ongoing process which shows the relevance of a language capable of unifying different areas of research. Probability theory for a long time has not been included among the basic mathematical tools of a physics curriculum but the situation is slowly changing and hopefully this will help cross fertilization among different disciplines.

Acknowledgements

This paper is an expanded version of a talk given at the meeting RG 2000, Taxco, Mexico, January 1999. It is a pleasure to thank C. Stephens and D. O'Connor for their kind invitation.

References

- [1] A.I. Khinchin, *Mathematical Foundations of Statistical Mechanics*, Dover Publications Inc., New York, 1949.
- [2] C. Di Castro, G. Jona-Lasinio, *Phys. Letts.* 29A (1969) 322.
- [3] K.G. Wilson, M.E. Fisher, *Phys. Rev. Letts.* 28 (1972) 240.
- [4] C. Di Castro, *Lett. Nuovo Cimento* 5 (1972) 69.
- [5] P.K. Mitter, *Phys. Rev.* D7 (1973) 2927.
- [6] E. Brezin, J.C. Le Guillou, J. Zinn Justin, *Phys. Rev.* D8 (1973) 434.
- [7] K.G. Wilson, *Phys. Rev.* B4 (1971) 3174.
- [8] K.G. Wilson, 140 (1965) B445.
- [9] K.G. Wilson, *Phys. Rev.* D2 (1970) 1438.
- [10] L.P. Kadanoff, *Physics* (N. Y.) 2 (1966) 263.
- [11] G. Jona-Lasinio, *Nobel Symp.* 24 (1973) 38.
- [12] G. Benettin, C. Di Castro, G. Jona-Lasinio, L. Peliti, A.L. Stella, in: M. Levy, P.K. Mitter (Eds.), *New Developments in Quantum Field Theory and Statistical Mechanics*, Plenum Press, New York, London, 1977.
- [13] M.E. Fisher, *Rev. Mod. Phys.* 70 (1998) 653.
- [14] A.Z. Patashinski, V.L. Pokrovski, *Fluctuation Theory of Phase Transitions* 2nd Edition, Nauka, Moskva, 1982 (in Russian).
- [15] P.M. Bleher, Ya.G. Sinai, *Comm. Math. Phys.* 33 (1973) 23.
- [16] G. Jona-Lasinio, *Nuovo Cimento B* 26 (1975) 9.
- [17] G. Gallavotti, G. Jona-Lasinio, *Comm. Math. Phys.* 41 (1975) 301.
- [18] Ya.G. Sinai, *Theor. Prob. Appl.* 21 (1976) 273.

- [19] G. Jona-Lasinio, in: M. Levy, P.K. Mitter (Eds.), *New Developments in Quantum Field Theory and Statistical Mechanics*, Plenum Press, New York, London, 1977.
- [20] P.M. Bleher, Ya.G. Sinai, *Comm. Math. Phys.* 45 (1975) 347.
- [21] P. Collet, J.P. Eckmann, *A Renormalization Group Analysis of the Hierarchical Model in Statistical Mechanics*, *Lecture Notes in Physics*, Vol. 74, Springer, Berlin, 1978.
- [22] M. Cassandro, G. Gallavotti, *Nuovo Cimento* 25B (1975) 691.
- [23] G. Gallavotti, H. Knops, *Rivista del Nuovo Cimento* 5 (1975) 341.
- [24] R.L. Dobrushin, in: R.L. Dobrushin, Ya.G. Sinai (Eds.), *Multicomponent Random Systems*, Dekker, New York, 1978.
- [25] R.L. Dobrushin, *Ann. Prob.* 7 (1979) 1.
- [26] Ya.G. Sinai, *Theory of Phase Transitions: Rigorous Results*, Pergamon Press, London, 1982.
- [27] G. Hegerfeldt, C. Nappi, *Comm. Math. Phys.* 53 (1977) 1.
- [28] M. Cassandro, G. Jona-Lasinio, in: L. Streit (Ed.), *Many Degrees of Freedom in Field Theory*, Plenum Press, New York, 1978.
- [29] I.A. Ibragimov, Yu.V. Linnik, *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff, Groningen, 1971.
- [30] M. Cassandro, G. Jona-Lasinio, *Adv. Phys.* 27 (1978) 913.
- [31] V.I. Oseledec, *Trans. Moscow Math. Soc.* 19 (1968) 197.
- [32] G. Jona-Lasinio, in: J. Frölich (Ed.), *Scaling and Self-similarity in Physics*, Birkhäuser, Basel, 1983.
- [33] P.M. Bleher, *Theor. Math. Phys.* 61 (1985) 1107.
- [34] G. Jona-Lasinio, *Helv. Phys. Acta* 59 (1986) 234.
- [35] A basic reference is G. Gallavotti *Rev. Mod. Phys.* 57 (1985) 471.
- [36] Yu.V. Linnik, I.V. Ostrovski, *Decomposition of Random Variables and Vectors*, *Translation of Mathematical Monographs*, AMS, Providence, RI, 1977.
- [37] D. Brydges, J. Dimock, T.R. Hurd, *Comm. Math. Phys.* 198 (1998) 111.
- [38] P.K. Mitter, B. Scoppola, *Comm. Math. Phys.* 209 (2000) 207.
- [39] G. Benfatto, G. Gallavotti, *Renormalization Group*, *Physics Notes*, Princeton University Press, Princeton, 1995.
- [40] G.F. De Angelis, G. Jona-Lasinio, V. Sidoravicius, *J. Phys. A* 31 (1998) 289.
- [41] M. Beccaria, C. Presilla, G.F. De Angelis, G. Jona-Lasinio, *Europhys. Lett.* 48 (1999) 243.
- [42] D. Brydges, I. Munoz Maya, *J. Theor. Prob.* 4 (1991) 371.