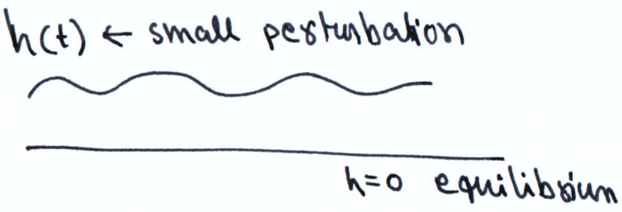
Linear response theory :



$$\langle m(t)\rangle_h - \langle m\rangle_0 = \int_{-\infty}^{t} ds\, \chi(t-s)\, h(s)$$

$$\chi(t) = -\beta \frac{d}{dt}\langle m(t)\, m(0)\rangle$$

⊛ h could be some other field, or pressure. In that case

$$\boxed{\chi(t) = -\beta \frac{d}{dt}\langle m(t)\, v(0)\rangle}$$

↳ is the ~~cong~~ thermodynamic conjugate variable of h.
[Ref. Book of Sethna]

What happens out-side equilibrium?

$$\chi(t) = -\frac{\beta}{2}\frac{d}{dt}\Big\{\langle m(t)\, v(0)\rangle + \langle m(0)\, v(t)\rangle\Big\}$$

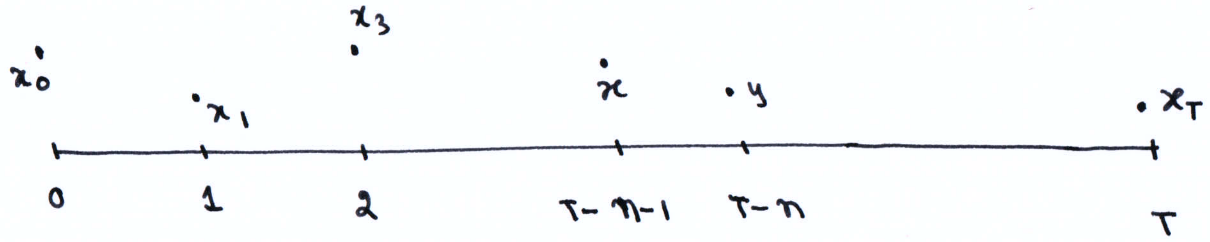
$$+ \Big\{\langle m(t)\, a(0)\rangle + \langle m(0)\, a(t)\rangle\Big\}$$

[activity]

Ref : TS & Derrida. 2016, J Stat mech, 113202
See last appendix.

Time-reversed process

[Book by Stroock
An Intro to Markov processes]



Forward evolution:

$$c(t) \equiv \{ c_0 = x_0,\ c_1 = x_1,\ \cdots\cdots,\ c_T = x_T \}$$

Prob evolves by $M(c', c)$

Time-reversed evolution: $c^*(t) = c(T-t)$

$$c^*(t) = \{ c_0^* = x_T,\ c_1^* = x_{T-1},\ \cdots\cdots,\ c_T^* = x_0 \}$$

Q: what dynamics describe $c^*(t)$?
Is it Markovian?

Answer:

$$P^*(c^*_{n+1} = x,\ c^*_n = y,\ \cdots,\ c^*_0 = x_T) = P(c_{T-n-1} = x,\ c_{T-n} = y,\ \cdots,\ c_T = x_T)$$

$$\Rightarrow M^*(c^*_{n+1} = x \mid c^*_n = y, \cdots, c^*_0 = x_T)\, P^*(c^*_n = y, \cdots, c^*_0 = x_T) = \cdots$$

Also

$$P^*(c^*_n = y, \cdots, c^*_0 = x_T) = P(c_{T-n} = y, \cdots, c_T = x_T)$$

Taking ratio

$$M^*\left(c^*_{n+1}=x \,\middle|\, c^*_n=y,\cdots,c^*_0=x_T\right)$$

$$= \frac{P\left(c_{T-n-1}=x,\, c_{T-n}=y,\,\ldots,\,c_T=x_T\right)}{P\left(c_{T-n}=y,\,\ldots,\,c_T=x_T\right)}$$

$$= \frac{P\left(c_{T-n-1}=x\right) M(x\to y)\, \cancel{M(y\to x_{T-n+1})}\cdots \cancel{M(x_{T-1}\to x_T)}}{P\left(c_{T-n}=y\right) \cancel{M(y\to x_{T-n+1})}\cdots \cancel{M(x_{T-1}\to x_T)}}$$

$$= \frac{P\left(c_{T-n-1}=x\right)}{P\left(c_{T-n}=y\right)} M(y,x)$$

$[M(x\to y) \equiv M(y,x)]$

This means that the time reversed dynamics is also Markovian (because it only depends on y, and not on $c^*_{n-1}, c^*_{n-2}, \cdots, c^*_0$).

But $M^*(x,y)$ depends on time (n) $\Longrightarrow$ a time inhomogeneous process.

Remark: If the forward process has reached a stationary state, i.e. $P(c_n=x) = P_{st}(x)$, then

$$\boxed{M^*(x,y) = \frac{P_{st}(x)}{P_{st}(y)} M(y,x)}$$

time-homogeneous.

$\left(\text{check: } \sum_x M^*(x,y) = 1\right)$

Remark: note, generally $M^*(x,y) \neq M(x,y)$. This equality happens only for equilibrium by detailed balance condition.

Remark: continuous time case can be obtained by

$$M^*(x,y) = \delta_{x,y} + \Delta t\, W^*(x,y) + \cdots = \frac{P(x)+\Delta t\, h(x)}{P(y)+\Delta t\, h(y)} \cdot \left(\delta_{y,x} + \Delta t\, W(y,x)\right)$$

5. Entropy and convergence to stationary state : [van-Kampen Book ch V, Sec 5]

A physics way of showing that system evolves towards its stationary state.

Similar to Boltzmann's H-theorem, (but simpler)

Let $f(x)$ is a strictly convex function, and non-negative

$$f(x) \geq 0 \quad \text{and} \quad f'(x) > 0 \quad \text{for} \quad x \in [0, \infty)$$

Let's define

$$H(t) = \sum_c P_{eq}(c)\, f\left(\frac{P_t(c)}{P_{eq}(c)}\right)$$

Assume that $P_{eq}(c)$ exist.

Then,

$$\frac{dH(t)}{dt} = \sum_c \cancel{P_{eq}(c)} \cdot f'(x_t(c)) \cdot \frac{1}{\cancel{P_{eq}(c)}} \cdot \frac{d}{dt} P_t(c)$$

$$= \sum_c f'(x_t(c)) \sum_{c'} \left\{ w(c,c')\, P_t(c') - w(c',c)\, P_t(c) \right\}$$

$$= \sum_{c'}\sum_c f'(x_t(c)) \cdot x_t(c') \cdot w(c,c')\, P_{eq}(c') - \underbrace{\sum_c \sum_{c'} f'(x_t(c)) \cdot x_t(c) \cdot w(c',c) \cdot P_{eq}(c)}_{\text{exchange } c \leftrightarrow c'}$$

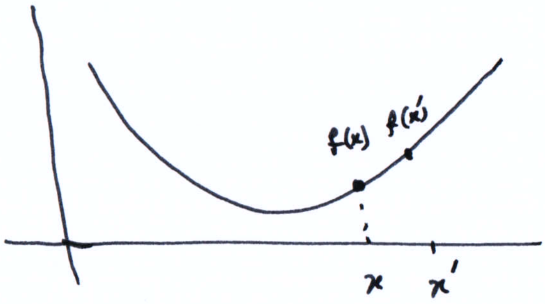
$$= \sum_c \sum_{c'} w(c,c') \cdot P_{eq}(c') \cdot x_t(c') \left[f'(x_t(c)) - f'(x_t(c')) \right]$$

use detailed balance

$$= \frac{1}{2} \sum_c \sum_{c'} w(c,c') \cdot P_{eq}(c') \cdot \left(x_t(c') - x_t(c) \right) \left(f'(x_t(c)) - f'(x_t(c')) \right)$$

$$\Rightarrow \frac{dH(t)}{dt} = -\frac{1}{2}\sum_{c}\sum_{c'} \omega(c,c')\cdot P_{eq}(c')\cdot\left(x_t(c)-x_t(c')\right)\left(f'(x_t(c))-f'(x_t(c'))\right)$$

≤ 0 for convex $f(x)$



This means, $H(t)$ monotonically decreases with time.

Let's choose $f(x) = x\log x$

Then $$H(t) = \sum_c \cancel{P_{eq}}(c)\cdot\frac{P_t(c)}{\cancel{P_{eq}}(c)}\cdot\log\frac{P_t(c)}{P_{eq}(c)}$$

$$= \sum_c P_t(c)\log\frac{P_t(c)}{P_{eq}(c)}$$ has minimum zero. (bounded below)

[Kullback-Leibler divergence]

$\Rightarrow$ As time grows, $H(t)$ could only monotonically decrease, then at $t\to\infty$ it can only reach $H=0$.

$$\Rightarrow \quad P_\infty(c) = P_{eq}(c).$$

<u>Remark</u>: You can use ~~the same~~ a very similar analysis to show that P_{eq} is unique.

This method is powerful, and typically used to prove unique asymptotic solution of ODE. One non-trivial example is solution of Hamilton-Jacobi equation $\frac{\partial S}{\partial t} = -H\left(q,\frac{\partial S}{\partial q}\right)$
[Dessida & TS: J. Stat. Phy. 177, 151 (2019)]

<u>Remark</u>: What happens if no detailed balance? What is non-equilibrium generalization of entropy? $\boxed{S(t) = -k_B H(t) + S_{eq}}$

More reading: Book of Sethna. ch 5.

⑥ If there is detailed balance, then we can ~~make~~ transform W-matrix symmetric, by a similarity transformation.

Then, we can use all nice properties of ~~Hermi~~ Hermitian matrices, just like in QM. (A very useful trick)

<u>How?</u>: use detailed balance

$$\omega(c',c)\, P_{eq}(c) = \omega(c,c')\, P_{eq}(c')$$

divide by $\sqrt{P_{eq}(c')P_{eq}(c)}$ $\Big\} \Rightarrow \frac{1}{\sqrt{P_{eq}(c')}} \cdot W(c',c) \cdot \sqrt{P_{eq}(c)} = \frac{1}{\sqrt{P_{eq}(c)}} \cdot W(c,c') \sqrt{P_{eq}(c')}$

$$\Rightarrow \boxed{H(c',c) = H(c,c')}$$

This means,

$$S^{-1} \cdot W \cdot S = H \quad \text{is a symmetric matrix.}$$

$$S = \begin{pmatrix} \sqrt{P(c_1)} & & 0 \\ & \sqrt{P(c_2)} & \\ 0 & & \ddots \end{pmatrix}$$

~~Hence, Eige~~

Symmetric matrix is diagonalizable and left eigenvector = right eigenvector

$$H|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle \qquad\qquad \langle\psi_\lambda|H = \lambda\langle\psi_\lambda|$$

$$\Rightarrow S^{-1} W S|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle \qquad\qquad \Rightarrow \boxed{\langle l_\lambda| = \langle\psi_\lambda|S^{-1}}$$

$$\Rightarrow W\left(S|\psi_\lambda\rangle\right) = \lambda\left(S|\psi_\lambda\rangle\right)$$

$$\Rightarrow \boxed{|r_\lambda\rangle = S|\psi_\lambda\rangle}$$

* Same eigen spectrum.

* Steady state ($\lambda = 0$) then corresponds to ground state of $-H$.

* Then any solution $|P_t\rangle$ can be expressed in terms of complete eigen states

$$|P_t\rangle = |P_{st}\rangle + e^{\lambda_1 t} S|\psi_1\rangle c_1 + e^{\lambda_2 t} S|\psi_2\rangle c_2 + \dots$$

with $0 > \lambda_1 > \lambda_2 > \dots$

This way, the time scale to reach stationary state is given by

$$\tau = \frac{1}{|\lambda_1|}$$

and this question of ~~having a~~ reaching a unique st state reduces to finding ~~any~~ non-zero spectral gap OR bound states for a QM hamiltonian.

A famous example:

Exclusion process $\longleftrightarrow$ XXX ~~XXXX~~ - spin chain

Deepak Dhar 1986.

Gwa & spohn, 1992.

⊛ Continuous time + continuous space.

$$\frac{d}{dt} P_t(x) = \int dy \left\{ \omega(x,y) P_t(y) - \omega(y,x) P_t(x) \right\}$$

$$\boxed{\frac{d}{dt} |P_t\rangle = W \cdot |P_t\rangle}$$

Generalization of Perron-Frobenius theorem

$\longrightarrow$ Perron-Frobenius-Jentzsch theorem

⊛ In most real examples, on "real" space, there are local jumps. $\Rightarrow$ at very small time dt, $\omega(x,y)$ decays very fast with $x-y$.

For such processes, the integral operator W can be reduced to a differential operator

$$\frac{d}{dt} P_t(x) = [\mathcal{L} \cdot P_t](x)$$

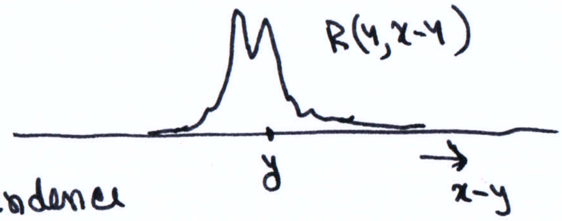
This equation is known as the <u>Fokker-Planck</u> equation.

This transformation is done by <u>Kramers-Moyal</u> expansion.

Kramers-Moyal expansion

$$\frac{d}{dt} P_t(x) = \int dy \left\{ W(x,y)\, P_t(y) - W(y,x)\, P_t(x) \right\}$$

let $$W(x,y) = R(\underset{\substack{\uparrow \\ \text{slow} \\ \text{dependence}}}{y}, \underset{\substack{\uparrow \\ \text{fast dependence}}}{x-y})$$



$$\Rightarrow \frac{d}{dt} P_t(x) = \int dy \left\{ R(y, x-y)\, P_t(y) - R(x, y-x)\, P_t(x) \right\}$$

$$= \int dy \left\{ R(x - \overline{x-y}, x-y)\, P_t(x - \overline{x-y}) - R(x, y-x)\, P_t(x) \right\}$$

$$= - \int dr \left\{ R(x-r, r)\, P_t(x-r) - R(x,-r)\, P_t(x) \right\}$$

$$= - \underbrace{\int dr \left\{ R(x,r)\, P_t(x) - R(x,-r)\, P_t(x) \right\}}_{0} + \frac{d}{dx}\left[a_1(x)\, P_t(x)\right] + \frac{d^2}{dx^2}\left[a_2(x)\, P_t(x)\right] - \dots$$

With $$a_n(x) = -\int dr \frac{(-r)^n}{n!} R(x,r)$$

$$\Rightarrow \boxed{\frac{d}{dt} P_t(x) = \frac{d}{dx}\left(a_1(x) P_t(x)\right) + \frac{d^2}{dx^2}\left(a_2(x) P_t(x)\right) + \dots}$$

$$= -\int dy \frac{(y-x)^n}{n!} W(y,x)$$

If we truncate at second order, it is called the FP equation.

<u>Remark</u>: this is usually justified if there is a scaling parameter (ℓ) such that $x-y \sim \ell$ small.

Example: Wiener process

$$R(x,r) = \frac{1}{\sqrt{2\pi \ell}} e^{-\frac{(r-\ell m)^2}{2\ell}}$$

[think CLT]

$$\Rightarrow \quad a_1(x) = + \, l \cdot m(x), \quad a_2(x) = - l + \cancel{x}\, l^2, \quad a_3 = l^2 + \cdots$$

Then, for small l,

$$\frac{d}{dt} P_t(x) = l \frac{d}{dx}(mP) - l \cdot \frac{d^2}{dx^2} P + \mathcal{O}(l^2)$$

$\boxed{\text{** One can absorb } l \text{ by redefining time } t \text{ **}}$

This, we will see is justified for Langevin equations, particularly for stochastic differential equations.

~~[illegible]~~ jumps

~~Another example: Ornstein-Uhlenbeck process~~

Remark: of course, there are cases where long jumps are allowed, one can not truncate the series.

for example Lévy process do not have a FP description.

Instead there is a description using ~~[illegible]~~ FP-type equation with $\boxed{\text{fractional derivatives.}}$

Ref! EPL, 46, 431 (1999).

Search in youtube channel "veubingx"

Remark: F-P equation, when written as

$$\frac{\partial P_t}{\partial t} = - \partial_x j \quad ; \quad j = - a_1(x) P - \frac{d}{dx}\left(a_2(x) P\right)$$

$j :=$ probability current.

~~In equilibrium $j(x,t) = 0$. Example: particle in a potential.~~

* Most common example

$$\frac{d}{dt} P_t(x) = \frac{d}{dx}\left[U'(x) P_t(x)\right] + k_B T \frac{d^2}{dx^2} P_t(x)$$

describes a Brownian particle in a potential $U(x)$ at temperature $K_B T$.

If $U(x)$ is a confining potential , system reaches an equilibrium stationary state, with zero current

$$U'(x) P_{eq}(x) + K_B T \frac{d}{dx} P_{eq}(x) = 0$$

$$\Rightarrow P_{eq}(x) \propto e^{-\frac{1}{K_B T} U(x)}$$

In operator formulation:

FP equation

$$\frac{\partial}{\partial t} P_t(x) = \mathcal{L} \cdot P_t(x) \qquad \text{time independent case.}$$

Here $\mathcal{L} \cdot P_t(x) = \frac{d}{dx} a_1(x) P_t(x) + \frac{d^2}{dx^2} a_2(x) P_t(x)$

is a differential operator.

Formal solution

$$P_t(x) = e^{t\mathcal{L}} \cdot P_0(x)$$

Remark: In general $\mathcal{L}$ is $\underbrace{\text{not hermitian (not self-adjoint}}_{\text{there is a subtle difference.}}$).

[If we define an inner product

$$\langle l | r \rangle = \int dx \, l^*(x) \, r(x)$$

Then, $\mathcal{L}^\dagger$ is defined by

$$\langle \mathcal{L}^\dagger \cdot l | r \rangle = \langle l | \mathcal{L} \cdot r \rangle$$

(a) $\mathcal{L}$ is Hermitian if $\mathcal{L}^\dagger \equiv \mathcal{L}$, meaning

$$\langle \mathcal{L} \cdot l | r \rangle = \langle l | \mathcal{L} \cdot r \rangle$$

(b) But, note that domain of $\mathcal{L}$ may not be same as domain of $\mathcal{L}^\dagger$.

functional space on which $\mathcal{L}$ acts.

If both domains are same, then $\mathcal{L}$ is self-adjoint.

This subtle difference has important consequence in QM.

An example: Consider $\mathcal{L} := \dfrac{d^2}{dx^2}$ acting on all functions on $0 \le x < \infty$ that vanish at $x \to \infty$ and

$$\int_0^\infty dx\, \ell(x)\, r(x) = \text{Finite}.$$

Then $$\langle \ell | \mathcal{L} \cdot r \rangle = \int_0^\infty dx\, \ell(x)\, r''(x) \qquad [\ell r'' = \ell'' r - (\ell r' - \ell' r)']$$

using integration by parts

$$= \int_0^\infty dx\, \ell''(x) \cdot r(x) + \underbrace{\ell(0)\, r'(0) - \ell'(0)\, r(0)}$$

vanish if

Case 1: $r(0) = 0 = r'(0)$

Case 2: $r(0) = 0 = \ell(0)$

Both cases give

$$\langle \mathcal{L}^\dagger \ell | r \rangle = \langle \ell | \mathcal{L} r \rangle = \int dx\, \ell''(x)\, r(x)$$

$$\Rightarrow \mathcal{L}^\dagger := \frac{d^2}{dx^2} = \mathcal{L} \quad \text{thus Hermitian.}$$

But in Case 1: ~~domain~~ space of functions $r \subseteq$ ~~domain~~ space of ℓ

$\Rightarrow$ domain of $\mathcal{L} \subseteq$ domain of $\mathcal{L}^\dagger$

so $\mathcal{L}$ is not self-adjoint

Case 2: both domains same $\Rightarrow \mathcal{L}$ is self-adjoint

Remark: This has interesting consequences in QM.

Ref 1: Araujo, Coutinho & Perez, Am. J. Phys. 72, 203 (2004)

Ref 2: Bonneau, Faraut, Valent, Am. J. Phys. 69, 322 (2001)

Generally, the Fokker-Planck operator is not self-adjoint.

- There will be left and right eigenvectors

$$\mathcal{L}\cdot r_\lambda(x) = \lambda\, r_\lambda(x) \qquad \mathcal{L}^{+}\cdot l_\lambda(x) = \lambda^{*}\, l_\lambda(x)$$

- Generalization of Perron-Frobenius theorem essentially applies when there is a 'spectral gap. ~~[illegible]~~ In such case,

(i) largest eigenvalue $= 0$

(ii) $r_0(x) = P_{st}(x)$

(iii) $l_0(x) = 1$.

- In good cases (most common) eigenbasis is

complete $\qquad \sum_\lambda l^{*}_\lambda(x)\, r_\lambda(x') = \delta(x-x')$

orthonormal $\qquad \int^{\infty} dx\, l^{*}_\lambda(x)\, r_{\lambda'}(x) = \delta_{\lambda,\lambda'}$

Then,

$$P_t(x) = r_0(x) + e^{t\lambda_1}\cdot r_1(x)\cdot a_1 + \dots$$

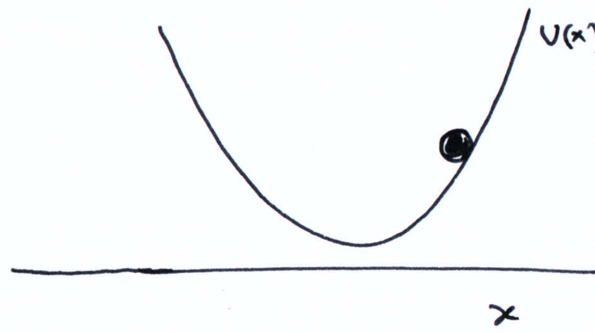
with $a_k = \int dx\, l^{*}_k(x)\, P_0(x)$

- Unlike in QM, here one needs to determine both the left and right eigenvectors.

⊛ **An explicit example:** Brownian particle in a potential $U(x)$.

$$\frac{d P_t(x)}{dt} = \mathcal{L} \cdot P_t(x)$$

$$\mathcal{L} := \frac{d}{dx} U'(x) + k_B T \frac{d^2}{dx^2}$$



~~Some physics intuition about space of functions~~

If $U \to \infty$ for $|x| \to \infty$, then our physics intuition tells that ~~we should~~ ~~look at functions~~ ~~which~~ $P_t(x)$, and its derivative vanish at infinity.

So we look at functions $r(x)$ such that

$$\overset{*}{l}(x)\, r(x) \longrightarrow 0 \quad \text{as} \quad |x| \to \infty$$
$$\overset{*}{l}(x)\, r'(x) \longrightarrow 0$$
$$\overset{*}{l}{}'(x)\, r(x) \longrightarrow 0$$

Then, for this space of functions (by integration by parts)

$$\mathcal{L}^+ := -U'(x) \frac{d}{dx} + k_B T \frac{d^2}{dx^2}$$

⊛ So $\mathcal{L}^+ \neq \mathcal{L} \Rightarrow$ not Hermitian.

~~Special case: $U(x) = \frac{1}{2}x^2$~~

~~Largest eigenvalue $\lambda = 0$, with~~

~~However, this problem can be fully solved~~

However, it can be made Hermitian by a similarity transformation.

↓

this links to Schrödinger equation.

F P equation $\longrightarrow$ Schrödinger equation

$$\frac{d}{dt} P_t(x) = \frac{d}{dx} U'(x) P_t(x) + K_B T \frac{d^2}{dx^2} P_t(x)$$

Define $P_t(x) = e^{-\frac{1}{2K_BT} U(x)} \psi_t(x)$

Do the algebra and show that

$$-\frac{d}{dt} \psi_t(x) = \left\{ -K_B T \frac{d^2}{dx^2} \psi_t + V(x) \psi_t \right\}$$

with effective potential

$$V(x) = \frac{1}{4K_BT} \left(U'(x) \right)^2 - \frac{U''(x)}{2}$$

Remark: what does it mean for $\mathcal{L}$-operator?

$$H = - \left(P_{eq}(x) \right)^{\frac{1}{2}} \mathcal{L} \left(P_{eq}(x) \right)^{-1/2} \qquad \left[P_{eq}(x) = e^{-\frac{U(x)}{K_BT}} \right]$$

a choice

$$= -K_B T \frac{d^2}{dx^2} + V(x) \quad \text{is } \underline{\text{Hermitian.}}$$

This makes thing easy.

① For H, both left and right eigenvectors are same.

② $\mathcal{L} \cdot r_\lambda = \lambda r_\lambda \Longrightarrow H \psi_\lambda = -\lambda \psi_\lambda$ with $r_\lambda(x) = \sqrt{P_{eq}(x)} \cdot \psi_\lambda(x)$

$\mathcal{L}^{+} \cdot l_\lambda = \lambda l_\lambda \longrightarrow l_\lambda(x) = \frac{1}{\sqrt{P_{eq}(x)}} \psi_\lambda(x)$

③ Eigenvalues λ are real.

④ Eigenvalue of $\mathcal{L}$ is minus of eigenvalue of H.

$\Rightarrow$ Steady state for $\mathcal{L}$ $\Leftrightarrow$ ground state of H.

This has important meaning:

Generalization of Perron-Frobenius, which essentially means that there is spectral gap between largest and second largest eigenvalues (non-degeneracy) $\Leftrightarrow$ question of spectral gap for H in QM $\Leftrightarrow$ existence of Bound states.

[Bound states in QM: ① Brownstein, Am. J. Phys. 68 (2000), 160
② Landau & Lifshitz]

An explicit solution

Ornstein-Uhlenbeck Process: Brownian particle in a harmonic potential. $U(x) = \frac{1}{2}x^2$

QM-potential

$$V(x) = \frac{x^2}{4K_BT} - \frac{1}{2}$$

Harmonic oscillator problem.

Eigenvalues $\lambda_n = -n$ with $n = 0, 1, 2 \ldots$

Eigen functions

$$\Psi_n(x) = \left[\frac{1}{2\pi K_BT}\right]^{1/4} \cdot \frac{1}{\sqrt{2^n n!}} \cdot H_n\left(\frac{x}{\sqrt{2K_BT}}\right) e^{-\frac{x^2}{4K_BT}}$$

Hermit polynomial.

$$\Rightarrow r_n(x) = \Psi_n(x) \cdot e^{-\frac{x^2}{4K_BT}}, \quad l_n(x) = \Psi_n(x)\, e^{+\frac{x^2}{K_BT}}$$

Show: $r_0(x) = P_{st}(x) \propto e^{-\frac{x^2}{2K_BT}}$ and $l_0(x) = 1$ (after trivial re-scaling)

Other examples: (A) Free brownian particle ($U(x) = 0$) on infinite line.

- Corresponding QM potential $V(x) = 0$:

It has only propagating solutions $e^{\pm ikx}$

with eigenvalues of $\mathcal{L}$-operator is continuous and given by

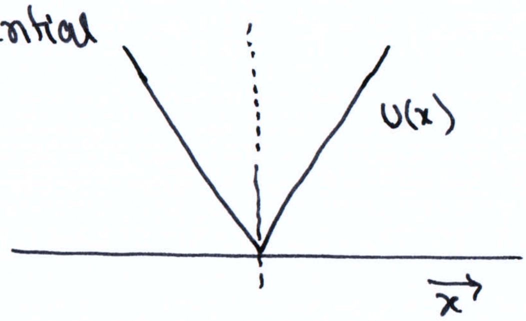
$$\lambda = -k^2 \cdot K_B T \quad \text{for} \quad k \geq 0.$$

- There is no gap in eigen-spectrum, and this means the Brownian particle takes infinite time to reach the uniform distribution.

Remark: do the same excercise on a ring, and show that the spectrum is discrete.

(B) Brownian particle in a linear potential

$$U(x) = |x|$$



$\Rightarrow$ QM potential

$$V(x) = \frac{1}{4 K_B T} - \delta(x)$$

There is only one bound-state, and that is the stationary state.

Spectral gap $= \frac{1}{4 K_B T}$