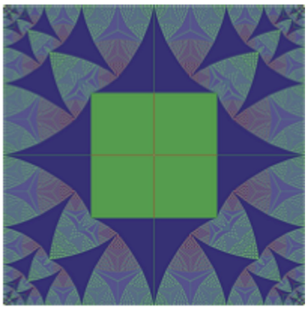
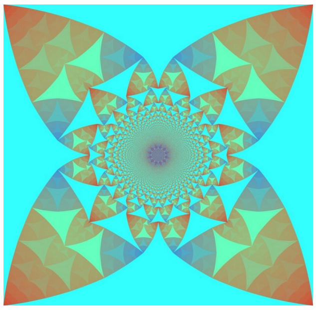
Third lecture

Tuesday, August 3, 2021 6:07 PM





[Picture taken from Lionel Levin]

Identity config on square grid.

Centrally fed growing sand pile.

Both have similar geometric structures. For now we study the second class of patterns.

Centrally fed growing sandpiles

On infinite square grid $\mathbb{Z}^2$. Threshold height $z_c = 4$.

2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2

Initial stable config.

drop grain at central node until it becomes unstable

2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	4	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2

relax

2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	3	2	2	2	2	2
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2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2

Repeat

2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
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2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2

relax

2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
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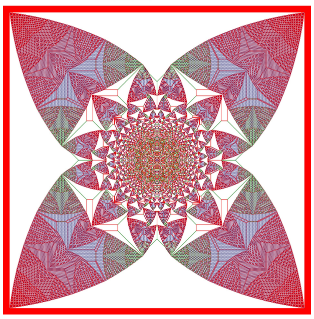
2	2	2	3	3	0	3	3	2	2	2
2	2	2	3	0	3	0	3	2	2	2
2	2	2	2	3	3	3	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2

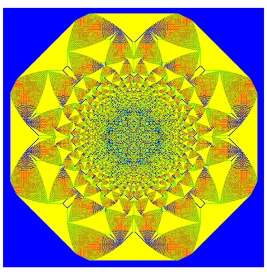
after many iterations

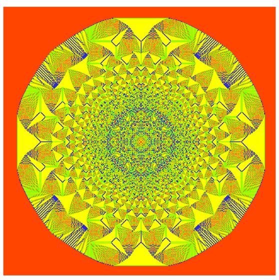


25×10^4 grains added

The pattern depends on the initial configuration.

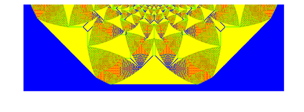
(different color code)



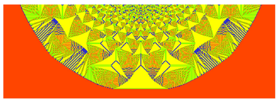


For $z_0 = 3$, avalanche does not stop.

For $z_0 \to -\infty$, the boundary of the pattern becomes a Euclidean circle.



Initial config $z_0 = 1$



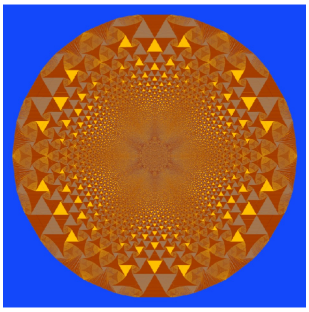
$z_0 = 0$

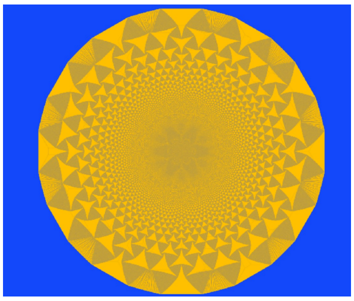
of the pattern becomes a Euclidean circle.

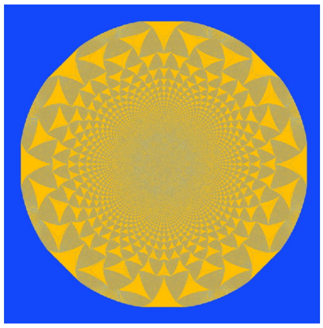
[Anne Fey ??]

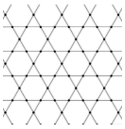
The pattern depends on the grid.

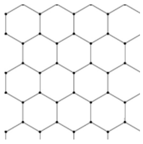
Initial height $z_0 = 0$

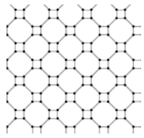






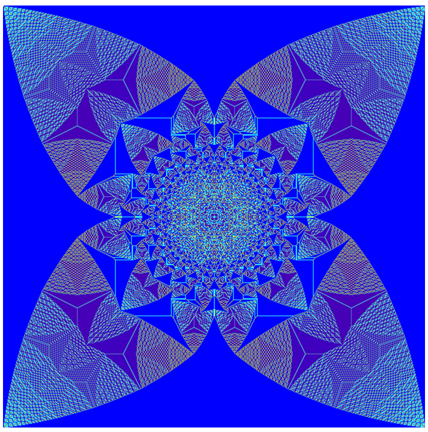




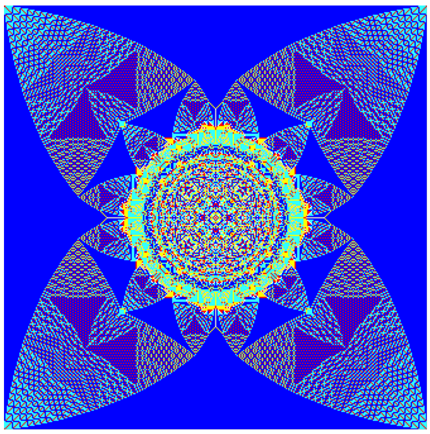


[Patterns taken from Wesley Pegden. For more examples of beautiful patterns see his website]

Higher dimension



Pattern on $\mathbb{Z}^2$
Initial height $z_0 = 2$

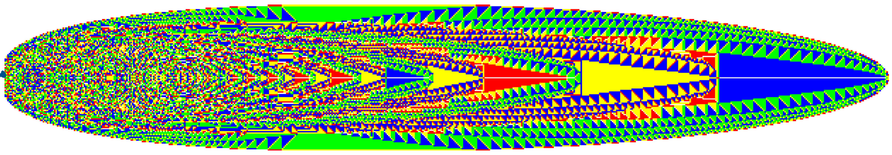


A slice of the pattern on $\mathbb{Z}^3$
Initial $z_0 = 4$

[Pattern taken from Lionel Levin]

Asymmetric toppling on $\mathbb{Z}^3$

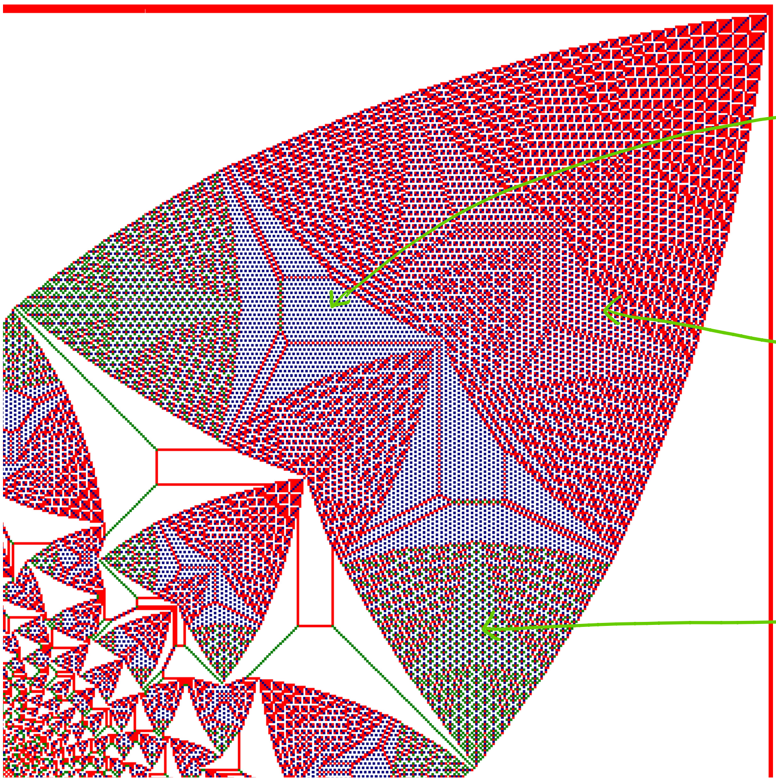
[Dhar and Sadhu, JSM (2013) P1106]



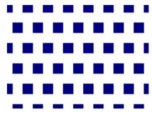
A 3d bug pattern

a cross section

Internal structure: In general patterns are made of patches inside which heights are periodic.

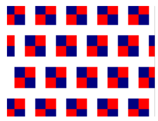


change in heigh in unit cell



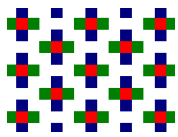
1	1
1	-2

$\Delta\rho = \frac{1}{4}$



0	-2	1
-2	0	1
1	1	1

$\Delta\rho = \frac{1}{9}$



		1		
	1	-2	1	
1	-2	0	-2	1
	1	-2	1	
		1		

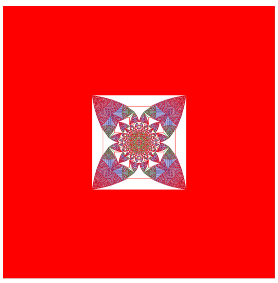
$\Delta\rho = \frac{1}{12}$

Other densities $\Delta\rho = \frac{1}{16}, \frac{1}{24}, \frac{1}{25}, \ldots$

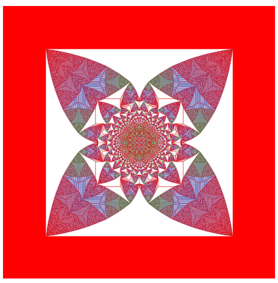
There is a large range of possible periodic arrangements inside patches. A classification of them can be found in an article [Srdjan Ostojic, Physica A 318 (2003), 187]

Boundaries of patches are sharp.

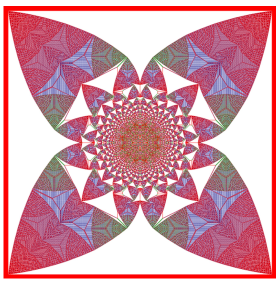
Proportionate growth [Dhar and Sadhu, JSM (2013) P11006]



5×10^4



2×10^5

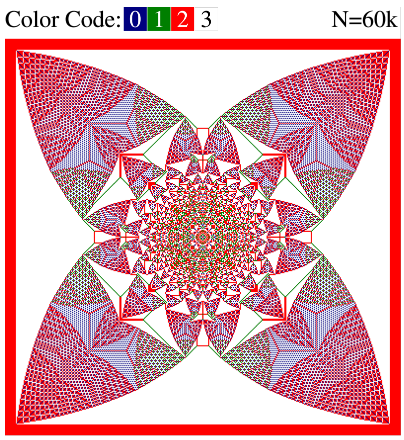


4×10^5

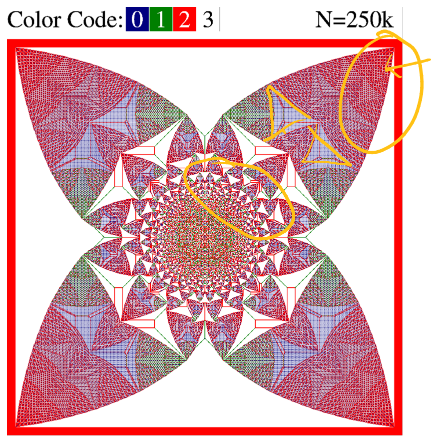
5×10^4 2×10^5 4×10^5

increasing number of added grains N

With increasing N, more and more internal structures are formed and once formed all grow keeping their relative size same such that the overall shape remains same. Patterning and growth happens together.



$N = 6\times10^4$

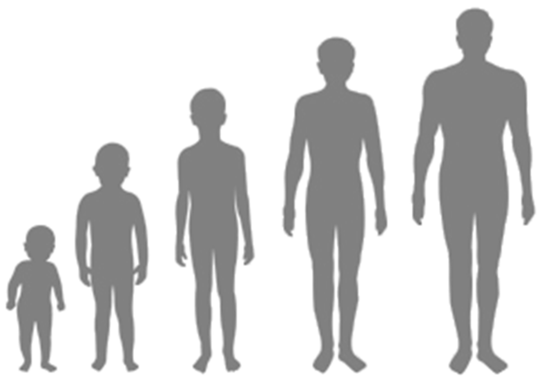


$N = 25\times10^4$

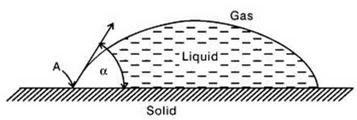
Pattern rescaled to be of same size.

This is called proportionate growth.

A motivation for us.



Animals grow in a highly coordinated fashion. It is a challenging problem in developmental biology to understand the basic mechanism for such growth.



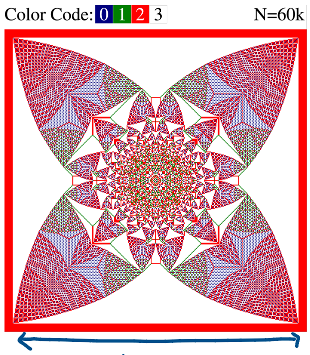
A simple example of proportionate growth in nature, but lacks complex internal structures.

How to construct a model where local dynamics generate a complex pattern which grows proportionately without a central regulation and also robust.

[conventional growth models like diffusion limited aggregation model, invasion percolation do not have these properties]

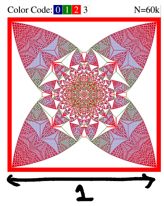
Characterization of the pattern

What we want to characterise.



Λ_N

$$(\xi,\eta) = \left(\frac{x}{\Lambda_N}, \frac{y}{\Lambda_N}\right)$$



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For larger N, the pattern grows and new internal structures emerge. As a result, for the rescaled pattern, size remains same but structure gets more resolved.
We want to characterize the asymptotic rescaled pattern in $N \to \infty$ limit.

Convergence of sandpile pattern.

Let $z_N(x,y)$ is height at site (x,y) for a pattern generated by adding N grains at the center $(0,0)$ and relaxing the configuration.

Let
$$\bar{z}_N(\xi,\eta) := z_N\left(\lfloor \xi \Lambda_N \rfloor, \lfloor \eta \Lambda_N \rfloor\right)$$

$\lfloor\ \rfloor$ = floor function.

Evidently there is no pointwise convergence for $\bar{z}_N(\xi,\eta)$ in $N \to \infty$ limit.
Instead, there is weak convergence.

> The bounded measurable function $\bar{z}_N(\xi,\eta)$ converges weakly as $N \to \infty$ limit,
> $$\bar{z}_N(\xi,\eta) \longrightarrow \rho(\xi,\eta)$$
> such that, for any continuous test function $\alpha(\xi,\eta)$ with compact support
> $$\int d\xi\, d\eta\; \bar{z}_N(\xi,\eta)\, \alpha(\xi,\eta) \longrightarrow \int d\xi\, d\eta\; \rho(\xi,\eta)\, \alpha(\xi,\eta)$$

[Precise theorem from "Convergence of the Abelian sandpile" Wesley Pegden & Charles Smart, Duke Math J, 162 (2013), 627.]

Theorem: [for ASM on $\mathbb{Z}^d$ with nearest neighbor toppling, no sink site, addition at center of initial empty background]

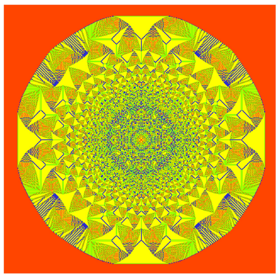
The rescaled sandpiles $\tilde{z}_N(x) := z_N(N^{1/d}x)$ converge weakly-* to a function $\rho \in L^\infty(\mathbb{R}^d)$ as $n \to \infty$. Moreover, the limit ρ satisfies $\int_{\mathbb{R}^d} \rho \, dx = 1$, $0 \le \rho \le 2d-1$, and $\rho = 0$ in $\mathbb{R}^d \setminus B_R$ for some $R > 0$.

space of bounded measurable functions on $\mathbb{R}^d$.

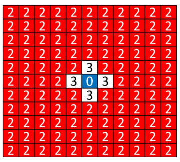
More comment: An upper bound on diameter of B_R is due to [Levin, Peres, Pot Anal 30 (2009)] which says that for every $\epsilon > 0$ there is a $c = c(\epsilon, d) > 0$ such that

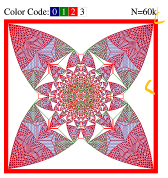
$$\{\tilde{z}_N > 0\} \subseteq B_{\left((d-\epsilon)|B_1|^{-\frac{1}{d}} N^{\frac{1}{d}} + c\right)}$$

for all $N > 0$. Here $|B_1| :=$ volume of unit ball.



For the example, with initial $z=2$ background it can be shown by induction that the boundary is square [Ostojic (2003)].





First note that boundary sites have height 3 and corner ones have height 2.
If an avalanch reaches any boundary site, then all boundary sites topple $\longrightarrow$ shifts the boundary by one unit to the same boundary height config.

Observing that the pattern is compact,

$$\Lambda_N = c_1 \sqrt{N} + \text{sub-leading in large } N.$$

[Fey, Levin, Peres (2010), JSP 138, 143]
[Fey, Meester, Redig (2007). Ann Prob, 37, 654]
"stability and percolation in infinite volume sandpiles"

Our work: (with Deepak Dhar)

We determine the $\rho(\xi, \eta)$ in terms of discrete holomorphic function.

Overall idea: on $\mathbb{Z}^2$.

[Ostojic, Physica A 318 (2003), 187]
[Dhar, Sadhu, Chandra, EPL, 85 (2009), 48002]

$T_N(x,y) :=$ number of toppling at site (x,y) in the pattern generated by adding N-grains.

[toppling function, Odometer function]

Conservation of particle number

$$\boxed{\Delta T_N(x,y) = z_N(x,y) - z_0(x,y) - N\,\delta_{x,0}\,\delta_{y,0}}$$

Rescaled variables

$$T_N(x,y) \simeq \Lambda_N^{\beta}\, \underline{\phi(\xi,\eta)} + \text{subleading in } \Lambda_N \gg 1.$$

$$z_N(x,y) \simeq \rho(\xi,\eta) + \text{subleading} \quad " \quad ".$$

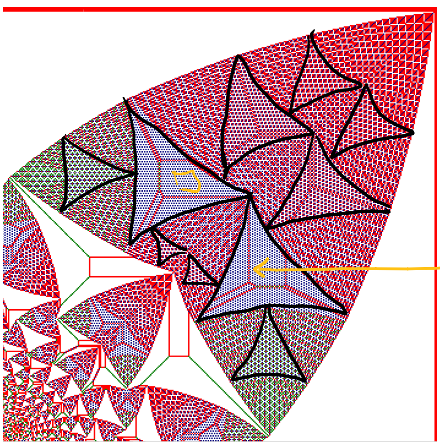
$$\Lambda_N^{\beta-2}\left(\partial_{\xi\xi} + \partial_{\eta\eta}\right)\phi(\xi,\eta) = \rho(\xi,\eta) - \rho_0 - \frac{N}{\Lambda_N^2}\,\delta(\xi)\,\delta(\eta)$$

For ASM on $\mathbb{Z}^2$, $\beta = 2$ and it gives

$$\boxed{\nabla^2\phi = \rho - \rho_0 - \lambda\,\delta(\vec{x})}$$

This is like an electrostatic problem: ϕ is the potential due to areal charge density $\rho-\rho_0$ and a point charge λ at origin. If we know $\phi(\vec{x})$, then we know the asymptotic pattern $\rho(\vec{x})$, which is our goal.

How do we know ϕ? For this we use a few basic features in the pattern.



(1) The asymptotic pattern is union of Patches inside which heights are periodic.

(2) Asymptotic density inside patches is constant.

(3) Defect lines contribute subleading order in areal density $\rho(\vec{x})$.

Proposition: Inside the periodic patches, $\phi(\xi,\eta)$ is atmost a quadratic polynomial.

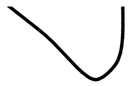
$$\phi(\xi,\eta) = a\xi^2 + b\eta^2 + c\,\xi\eta + d\xi + e\eta + f$$

Argument: Inside a patch, ϕ is Taylor expandable

$$\phi(\xi_0+\Delta\xi,\ \eta_0+\Delta\eta) - \phi(\xi_0,\eta_0)$$

$$= a_1\Delta\xi + b_1\Delta\eta + a_2(\Delta\xi)^2 + b_2(\Delta\eta)^2 + h_2\,\Delta\xi\,\Delta\eta$$

$$+\ a_3(\Delta\xi)^3 + \cdots$$



$$= a_1 \Delta\xi + b_1 \Delta\eta + a_2 (\Delta\xi)^2 + b_2 (\Delta\eta)^2 + h_2 \Delta\xi \Delta\eta + \underline{a_3 (\Delta\xi)^3} + \cdots$$

This relates to integer toppling function

$$T_N(x,y) = \Lambda_N^2 \, \phi(\xi,\eta) + \text{subleading in } \Lambda_N$$

$$\Rightarrow \quad T_N(x_0+\Delta x, y_0+\Delta y) - T_N(x_0,y_0) = \Lambda_N a_1 \Delta x + \Lambda_N b_1 \Delta y + a_2(\Delta x)^2 + b_2 (\Delta y)^2 + h_2 \Delta x \Delta y + a_3 \frac{(\Delta x)^3}{\Lambda_N} + \cdots$$

Because $T_N(x,y)$ is an integer valued function, the difference jumps discontinuously at every $\underline{\Delta x \sim \Lambda_N^{1/3}}$ due to the $(\Delta x)^3$ term.
This would then mean that there are large number of defect lines at distances $\sim \Lambda_N^{1/3}$ in a patch of size $\sim \Lambda_N$.
This we don't see in the pattern, and therefore any term of degree 3 or higher must be absent in $\phi(\xi,\eta)$.

How does it help us determine ϕ?

- Rescaled toppling function in the asymptotic patch is a piece-wise quadratic function.
- Inside each patch, we need to determine five parameters in
$$\phi(\xi,\eta) = a\xi^2 + b\eta^2 + c\,\xi\eta + d\xi + e\eta + f$$
- Continuity of ϕ and its first derivatives give constraints on these parameters.
- Solve these constraints with boundary condition $\phi = 0$ out-side the pattern, and logarithmic divergence near the centre.

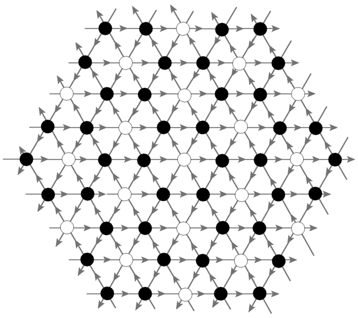
The constraints are hard to solve in a general pattern. However, it is possible to explicitly solve these constraints in certain examples. The square grid ASM has a large range of different kinds of patches and rather complex to begin with. We (ideas of Deepak Dhar) constructed relatively simpler models where the above constraints can be solved explicitly.

Example 1: Pattern on a directed triangular lattice. [Sadhu, Dhar, PRE, 85, 021107 (2012)]

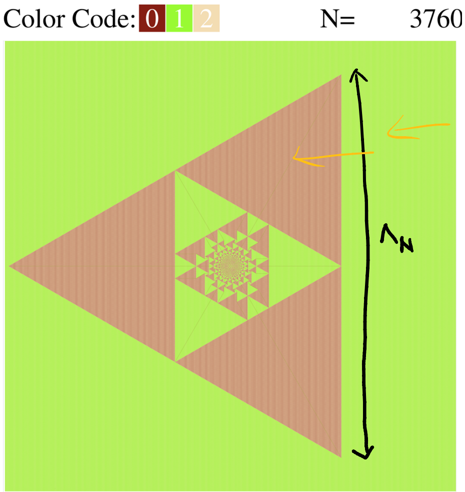
● = 1

○ = 2

$\rho_0 = \frac{4}{3}$



Threshold height $z_c = 3$



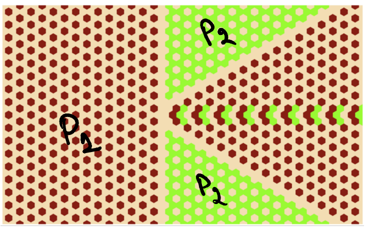
Properties:

(1) These are fast-growing patterns $\Lambda_N \sim N$ for large N.

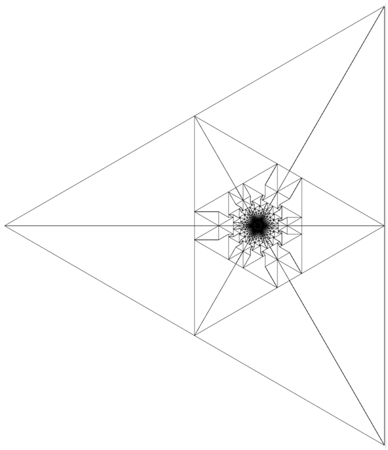
Added grains get distributed along 1d curves.

(2) There are only two kinds of patches, both of same areal density equal to the background density.

(3) Patch boundaries are straight lines of slopes $0, \frac{\pi}{6}, \frac{2\pi}{6}, \dots, \frac{5\pi}{6}$.

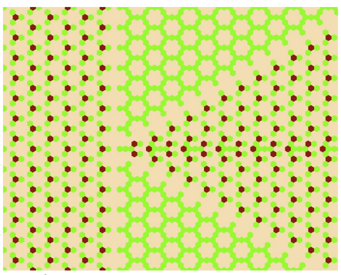


Both patches P_1 and P_2 have same unit cell density



Plot of difference of unit cell density with respect to background density

$\Delta\rho = \rho - \rho_0$



Periodicity in a similar pattern on a different background.

Added grains accumulate along the patch boundaries denoted by black straight lines. Along these lines average change in height $-\frac{1}{\sqrt{3}}, 1, \frac{2}{\sqrt{3}}$

(a) Equation for the rescaled toppling function.

Following our earlier arguments

$$\Lambda_N^{\beta-2}\left(\partial_{\xi\xi}+\partial_{\eta\eta}\right)\phi(\xi,\eta) = \rho(\xi,\eta) - \rho_0 - \frac{N}{\Lambda_N^2}\delta(\xi)\delta(\eta)$$

$$\Lambda_N \left(\partial_{\xi\xi} + \partial_{\eta\eta}\right) \phi(\xi,\eta) = \left(\rho(\xi,\eta) - \rho_0\right) - \frac{\lambda}{\Lambda_N^2}\,\delta(\xi)\,\delta(\eta)$$

$\beta = 1$

$\Rightarrow\ T_N \simeq \Lambda_N \phi$

$\dfrac{\sigma(\xi,\eta)}{\Lambda_N}$ $\qquad \dfrac{\lambda}{\Lambda_N}$

$$\boxed{\nabla^2 \phi(\bar{r}) = \sigma(\bar{r}) - \delta(\bar{r})}$$

$\bar{r} \equiv (\xi,\eta)$

$\sigma(\bar{r}) \equiv$ line charge density.

(b) Inside each patch ϕ is linear

$$\phi_p(\bar{r}) = d_p\,\xi + e_p\,\eta + f_p$$

So only three parameters to determine.

(c) Conditions at the boundary of these patches.

$$\phi_p = \phi_{p'} \qquad \text{and} \qquad \hat{n}\cdot\nabla(\phi_p - \phi_{p'}) = \sigma$$

$\hat{n}$ unit vector perpendicular to patch boundary.

Remark: For this pattern, it turns out that $T_N(\bar{R})$ is itself linear inside patches.

$$T_N(\bar{R}) = A_p + \bar{K}_p\cdot\bar{R} + F_{\text{Periodic}}(\bar{R})$$

Then, if $\hat{e}_1, \hat{e}_2$ are basis vectors of unit cell of the periodicity inside the patch,

$$T_N(\bar{R} + \hat{e}_{1(2)}) - T_N(\bar{R}) = \bar{K}_p\cdot\hat{e}_{1(2)}$$

As T_N is integer valued, $\bar{K}_p\cdot\hat{e}_{1(2)} =$ integer. If $\hat{g}_1, \hat{g}_2$ are reciprocal vectors

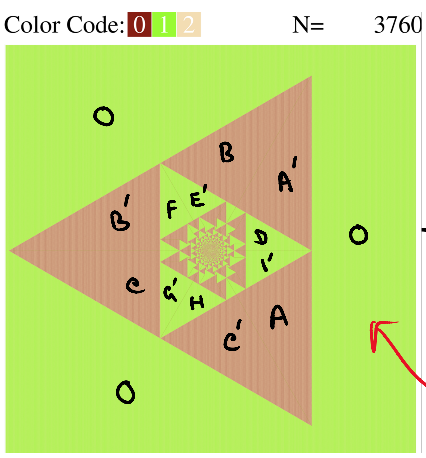
$$\hat{g}_i\cdot\hat{e}_j = \delta_{ij}$$

then

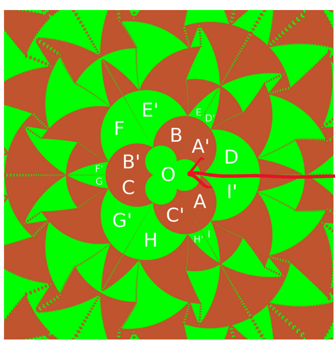
$$\bar{K}_p = m\,\hat{g}_1 + n\,\hat{g}_2 \qquad \text{with } (m,n) \text{ integers.}$$

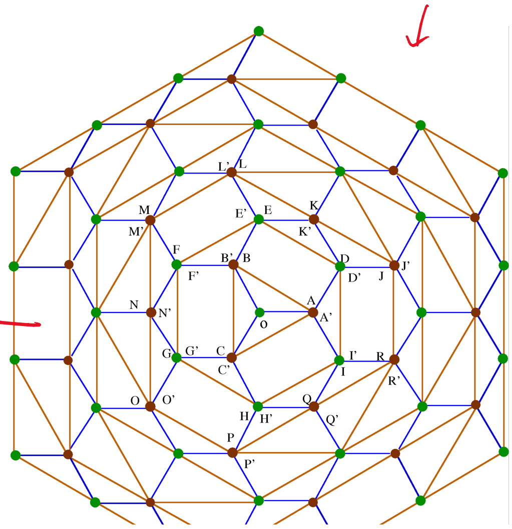
For our example, $\hat{e}_1, \hat{e}_2$ form triangular lattice $\Rightarrow$ $\hat{g}_1, \hat{g}_2$ form hexagonal lattice.

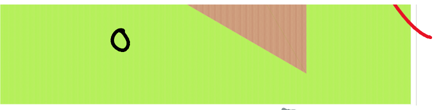
(d) How to order the patches? Adjacency graph.

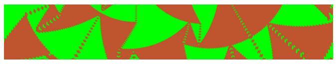


$\xrightarrow{}$ $1/\sqrt{2}$ transformation.

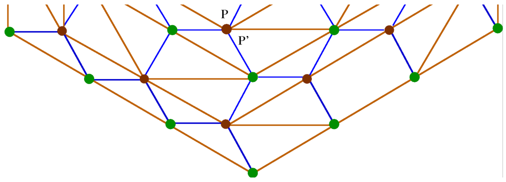








[Observe how patches (A, A') are treated as same node. Same for other pairs.]

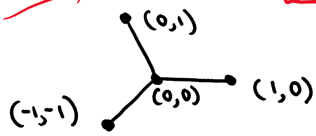


Adjacency graph

On the adjacency graph :

(1) Each node position can be denoted by

$$D = m + n\,\omega \qquad \text{with} \quad \omega \equiv \text{cube root of unity.}$$
$$= e^{i2\pi/3}$$



and $\quad 1+\omega+\omega^2 = 0$

Therefore each patch is associated with a pair of integers (m,n).

(2) If we consider the matching conditions

$$\hat{n}\cdot\nabla(\phi_p - \phi_{p'}) = \sigma \qquad \text{with} \quad \sigma = -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$$

along the patch boundaries we see that

$$\boxed{\phi_p = -\frac{1}{2\sqrt{3}}\left[D_p\bar{z} + \bar{D}_p z\right] + f_p \qquad \text{with} \quad p \equiv (m,n). \quad z = \xi + i\eta}$$

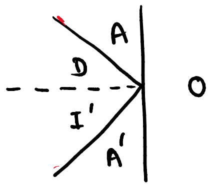
This gives the linear order terms.

(3) How do we get f_p ? Use the second matching condition $\phi_p = \phi_{p'}$ along patch boundaries.

Across a patch boundary $z = re^{i\theta} + u$ it gives

$$\boxed{f_{p'} - f_p = \mathrm{Re}\left[\bar{u}\,(D_{p'} - D_p)\right]\cdot\frac{1}{\sqrt{3}}}$$

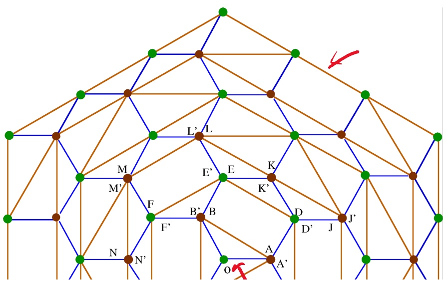
we still need additional constraint. This comes from concurrency condition. For example, patch boundaries OA, DA, I'A' meet at same point, ie, same u. Then the above condition for f_p gives



$$f_O + f_D + f_{I'} = 3 f_A$$

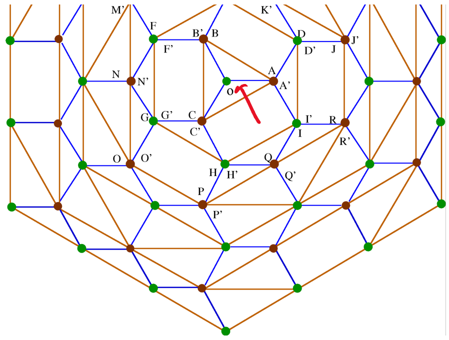
Extending this condition on all nodes on the adjacency graph gives that

$\boxed{\Delta f = 0}$ on the hexagonal graph formed by blue edges.



[continuity condition along orange edges

[continuity condition along orange edges are automatically satisfied]



(4) Then f_p is discrete harmonic on hexagonal graph with $f=0$ for $(0,0)$ node which corresponds to outside of the pattern.

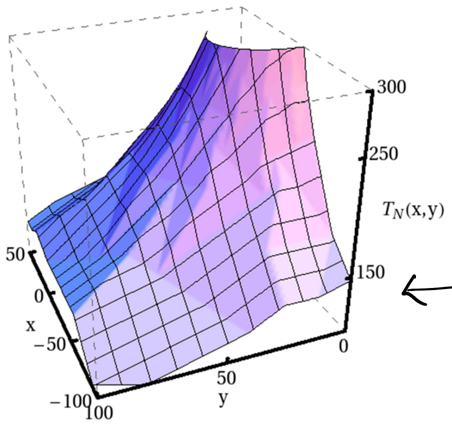
Solution

$$f_{mn} = \frac{c}{4\pi^2}\int_{-\pi}^{\pi}\int dk_1\, dk_2 \; \frac{1-\cos\left(\frac{k_1(2m-n)}{3}+k_2 n\right)}{1-\frac{\cos 2k_2+2\cos k_1\cos k_2}{3}}$$

(5) How do we get c?

$$\nabla^2\phi = \sigma - \lambda\,\delta(r)$$

$$\Rightarrow \phi(\vec{r}) \simeq -\frac{\lambda}{2\pi}\log|r| \quad \text{in 2d}$$



The ϕ-function is a piece-wise linear approximation to this logarithmic function.

The piece-wise linear toppling function (only a section is shown).

This gives (see PRE 85, 021107 (2012))

$$\Rightarrow \boxed{f_p \simeq \frac{\lambda}{2\pi}\log|m+n\omega|} \quad \text{for large } m,n.$$

[More precisely it comes from a reasoning that there are complex coordinates r_0 inside each patch (m,n) with $|m|+|n|$ large, where

$$\frac{\partial}{\partial \vec{r}}\left[\phi(r)+\psi(r)\right]\Big|_{r_0} = 0 \qquad \text{and} \qquad \phi(r_0)=\psi(r_0) \quad \text{where } \psi(r) = -\frac{1}{2\pi}\log|r|$$

← Keep this relation in mind as it will appear again]

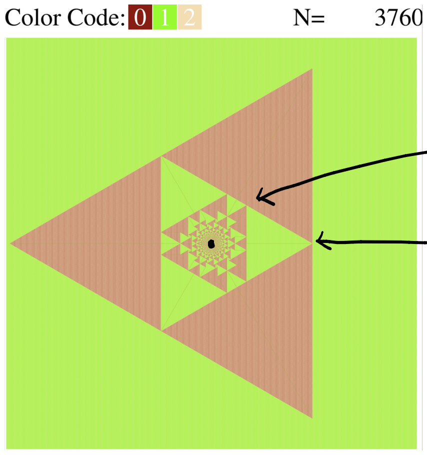
$$\boxed{\text{This gives} \quad c = \frac{\lambda}{\sqrt{3}}.}$$

What do we get?

Knowing D and f determines ϕ completely as piece-wise linear functions.

We can construct the patch boundaries from this solution.

For example,



$\left(\frac{2}{\sqrt{3}} - \frac{3}{\pi}, -\frac{1}{3} + \frac{\sqrt{3}}{\pi}\right)$

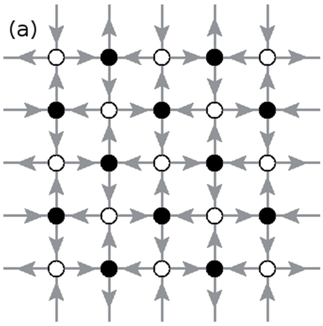
$\left(\frac{1}{\sqrt{3}}, 0\right)$

Example 2:

● = 1

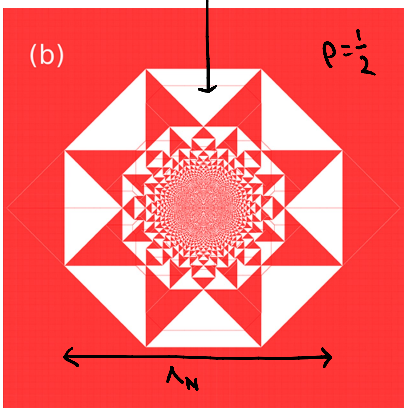
○ = 0

$\rho_0 = \frac{1}{2}$



F - lattice

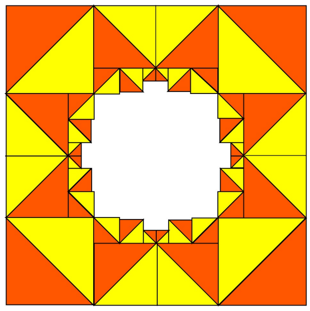
Threshold height $z_c = 2$



$\rho = 1$

There are two types of patches $\rho = \frac{1}{2}$ and $\rho = 1$

There is a correspondance with "squaring the square" a la Brooks - Smith - Stone - Tutte (ask Deepak for details)

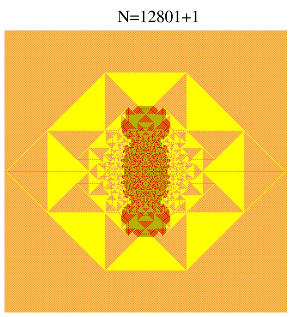


Pattern made of square tilings.

gives coeff 1.

Growth of the boundary

$$\Lambda_N \simeq \sqrt{N} + O(N^{1/4}) \quad \text{for large } N.$$



Shaded dark region shows toppled region in one relaxation event. Only when one such avalanch reach boundary, the pattern grows in size.

Rescaled toppling function

$$T_N(x,y) \simeq \Lambda_N^2 \, \phi(\xi, \eta) \quad \text{for large } N.$$

It follows
$$\nabla^2 \phi = \rho - \rho_0 - \lambda \delta(\bar{r})$$

The ϕ function is piece-wise quadratic.

$$\phi = \frac{1}{8}(1+m)\xi^2 + \frac{1}{4} n \xi\eta + \frac{1}{8}(1-m)\eta^2 + d\xi + e\eta + f \quad \text{in patch } \rho = 1.$$

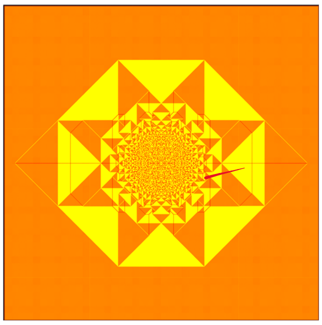
$$\phi = \frac{1}{8} m \xi^2 + \frac{1}{4} n \xi\eta - \frac{1}{8} m \eta^2 + d\xi + e\eta + f \quad \text{in patch } \rho = \frac{1}{2}.$$

Matching conditions along patch boundaries.

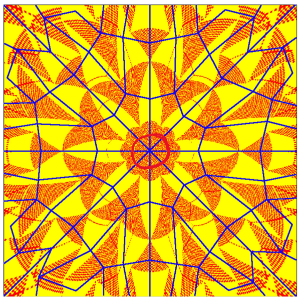
Matching conditions along patch boundaries.

$$\hat{n}\cdot\bar{\nabla}\left(\phi_p - \phi_{p'}\right) = 0 \qquad \text{and} \qquad \phi_p = \phi_{p'}$$

To solve the constraints, arrange the patches

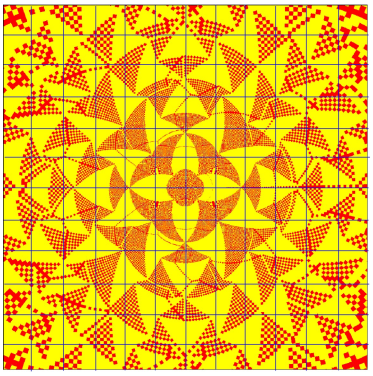


$1/\bar{z}$ transformation $\longrightarrow$

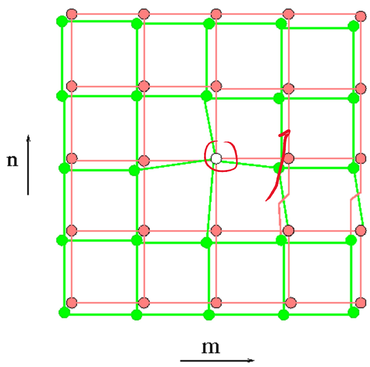


$1/\bar{z}^2$ transform $\downarrow$

a grid on two sheeted Riemann surface. $\downarrow$



$\Longleftrightarrow$



Following the matching condition

$$\hat{n}\cdot\nabla\left(\phi_p - \phi_{p'}\right) = 0$$

we see that (m,n) are integer indices of the square grid on a doubly sheeted Rieman surface.

How do we determine rest of the coefficients d, e, and f ?

Use the second matching condition $\phi_p = \phi_{p'}$

It gives $\boxed{D_p = d_p + i e_p}$ as discrete holomorphic function on the adjacency graph, ie it follows discrete Cauchy-Riemann condition except at origin.

As consequence, D_p is discrete harmonic.

$\Delta D_p = 0$ on two sheeted Riemann Surface with branch point at origin.

Boundary conditions

at origin.

Boundary conditions

$D_p = 0$ for $(0,0)$ node (because no toppling outside pattern)

$$\boxed{D_p \equiv D_{m,n} = \pm \frac{\lambda}{\sqrt{2\pi}} \sqrt{m+in}}$$ for large m,n. (because of logaritmic divergence of ϕ near centre of the pattern)

($\pm$ are for two sheets.)

Then D is discrete analogue of $\sqrt{z}$.

The coefficient f_p is expressed interms of D_p following the matching condition.

A few simple consequences:

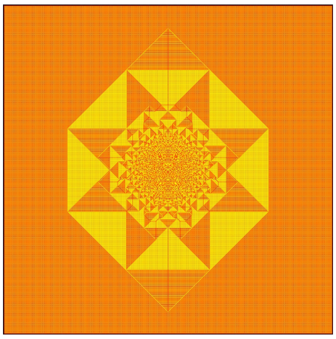
(1) patch boundaries can be determined.

(2) We can show that the pattern has eight-fold symmetry.

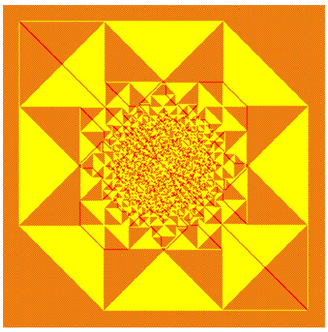
(3) Number of patches $n(A)$ with area A scales as

$$n(A) \sim 1/A^{5/3}$$

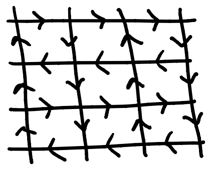
Remark: There are more than one backgrounds which gives same asymptotic pattern. Also different lattices can give same pattern.



on F-lattice with different background

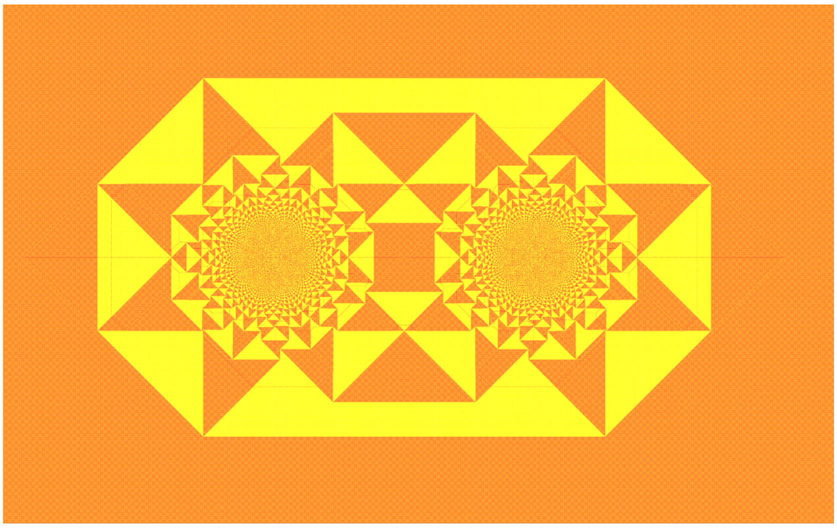


on Manhattan lattice

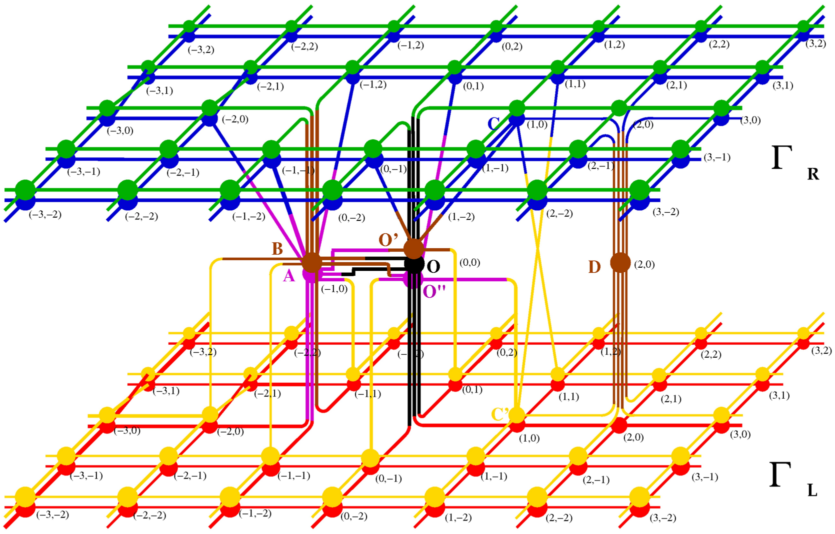


Manhattan lattice.

Example 3: Multiple addition sites.

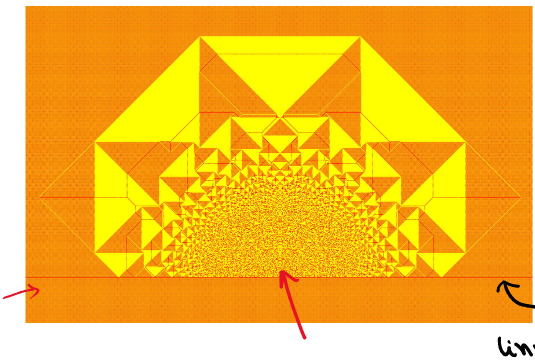


grains are added at two points at separation $\Delta x = \alpha \cdot \sqrt{N}$ on the F-lattice.

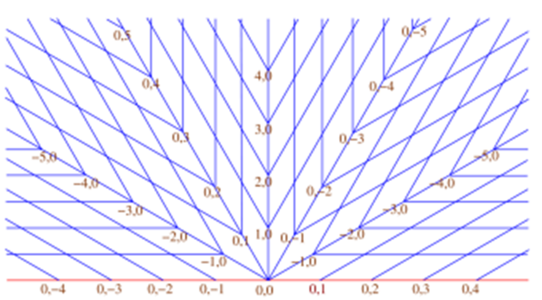


Adjacency graph.

Example 4: Absorbing lines



line of sink sites.

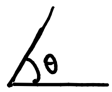


Adjacency graph

(1) $\Lambda_N \sim N^{1/3}$

Other geometries of sink sites

(2) Inside a wedge of sink lines of angle θ



$$\Lambda_N \sim N^{\alpha} \quad \text{with} \quad \alpha = \frac{1}{2+\frac{\pi}{\theta}}$$

(3) If the sink site is adjacent to the addition site

$$\Lambda_N \sim \sqrt{\frac{N}{\log N}}.$$

Example 5: Intermediate growth rate is possible even without a sink site.