Ising by spin chain: for print

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1d Ising to quantum spin chain.

Ref: ch 2.2 of the book of Mussardo.

Partition function (periodic chain):

$$Z_L = \sum_{\{\sigma_i\}} \prod_{i=1}^{L} e^{J\sigma_i\sigma_{i+1}} = \underbrace{Tr\left(T^L\right)}_{\text{transfer matrix}} \qquad \text{with } T(\sigma',\sigma) = e^{J\sigma'\sigma}$$

$$= \begin{array}{c|cc} & |+\rangle & |-\rangle \\ \hline \langle +1 & e^{J} & e^{-J} \\ \langle -1 & e^{-J} & e^{J} \end{array}$$

The space of 2×2 matrices can be expressed in the basis formed by

$$\mathbb{1} = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \text{ and pauli matrices } \hat\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}, \hat\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \hat\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

Using the transfer matrix in this basis we get

$$T = e^{H} \quad \text{with } H = e^{-2J}\,\hat\sigma_x + \tfrac{1}{2}\log(\sinh 2J)$$

This way partition function $Z_L = Tr\left(e^{LH}\right)$ for a 1d classical Ising model

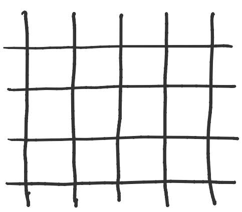
$$= \sum_{\sigma} \langle\sigma| e^{LH} |\sigma\rangle$$

maps to quantum evolution (upto wick's rotation) of a single site problem (0 dimension)

Exercise: Using invariance of trace under change of basis, derive the Ising 1d result we got earlier.

Ising model on a square lattice (2d)

[A solution due to Lieb-Shultz-Mattis
Rev Mod Phys 36 (1964), 856]



$\mu_1 \quad \mu_2 \quad \mu_3 \qquad \mu_L$

$$\mu_i = \{\sigma_{i,1}, \sigma_{i,2}, \cdots, \sigma_{i,L}\} \qquad 2^L \text{ vector} \equiv \{(\uparrow\downarrow), (\uparrow\downarrow), \cdots (\uparrow,\downarrow)\}$$

We shall write a transfer matrix between columns

$$\underbrace{T(\mu',\mu)}_{2^L\times 2^L} = e^{E(\mu',\mu) + E(\mu')}$$

$E(\mu',\mu)$: Energy from interactions between μ' and μ column.

$E(\mu')$: energy from interactions inside the μ' column

Then,

$$\boxed{Z_{L\times L} = Tr\left(T^L\right)}$$

Next decomposition:

$$T(\mu',\mu) = e^{E(\mu')} \cdot e^{E(\mu',\mu)}$$

$$= \sum_{\mu''} e^{E(\mu')}\, \hat\delta_{\mu',\mu''} \cdot e^{E(\mu'',\mu)}$$

$$= \sum_{\mu''} \underbrace{e^{E(\mu')}\,\hat{\delta}_{\mu',\mu''}}_{V_2(\mu',\mu'')} \cdot \underbrace{e^{E(\mu'',\mu)}}_{V_1(\mu'',\mu)}$$

$$V_2(\mu',\mu'') \to \left(\prod_{k=1}^{L} e^{J\sigma'_k\sigma'_{k+1}}\right)\delta_{\sigma'_1,\sigma''_1}\cdots\delta_{\sigma'_L,\sigma''_L}$$

$$V_1(\mu'',\mu) \to \prod_{k=1}^{L} e^{J\sigma''_k\sigma_k}$$

A better compact form: V_1 and V_2 are $2^L\times 2^L$ dimensional matrix. If we define matrices $\tilde{\sigma}_i^x = \underbrace{\mathbb{1}\otimes\mathbb{1}\otimes\cdots}_{2\times 2 \text{ matrices}}\otimes\tilde{\sigma}_i^x\otimes\cdots\otimes\mathbb{1}$ and similarly for σ^y, σ^z,

we can write

$$T = \prod_{i=1}^{L} e^{J\tilde{\sigma}_i^z\tilde{\sigma}_{i+1}^z + J\tilde{\sigma}_i^x}$$

Now, we want write as $T = e^H$. Because $\tilde{\sigma}^z, \tilde{\sigma}^x$ do not commute we need to use Baker-Campbell-Hausdorf formula.

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}\left([[A,B],B]+[A,[A,B]]\right)+\cdots}$$

Then using

$$[\tilde{\sigma}_i^\alpha, \tilde{\sigma}_j^\beta] = \delta_{ij}\,\epsilon_{\alpha\beta\gamma}\,(2i)\,\tilde{\sigma}_i^\gamma$$

we arrive at

$$\boxed{T = e^H \quad \text{with} \quad H = \sum_{i=1}^{L}\tilde{\sigma}_i^x + e^{2L}J\sum_{i=1}^{L}\tilde{\sigma}_i^z\tilde{\sigma}_{i+1}^z}$$

This way the 2-d classical Ising model partition function reduces to an 1-d quantum spin chain.

For the Ising partition function, $Z = \mathrm{Tr}(T^L)$ which is invariant unitary transformation or change of basis. It is convenient (for next calculations), to consider

$$H = \sum_{i=1}^{L}\sigma_i^z + \underbrace{e^{2L}J}_{\lambda}\sum_{i=1}^{L}\sigma_i^x\sigma_{i+1}^x$$

Solving this problem by mapping to free fermion problem:

Step 1: Write in terms of creation-annihilation operators.

$$\sigma_i^{\pm} = \frac{1}{2}\left[\sigma_i^x \pm i\sigma_i^y\right] \qquad \text{Creation-annihilation operators.}$$

In terms of this $\boxed{\sigma_i^x = \sigma_i^+ + \sigma_i^- \quad \text{and} \quad \sigma_i^z = \sigma_i^+\sigma_i^- - \frac{1}{2}}$

In terms of this $\sigma_i^x = \sigma_i^+ + \sigma_i^-$ and $\sigma_i^z = \sigma_i^+\sigma_i^- - \frac{1}{2}$

Commutation relations, $[\sigma^x, \sigma^y] = 2i\sigma^z$, $[\sigma^y, \sigma^z] = 2i\sigma^x$, $[\sigma^z, \sigma^x] = 2i\sigma^y$

$\{\sigma^\alpha, \sigma^\beta\} = 2\delta_{\alpha\beta}$,

$[\sigma^+, \sigma^-] = 2\sigma^z$, $[\sigma^z, \sigma^\pm] = \pm\sigma^\pm$

$\{\sigma^+, \sigma^-\} = -1$ and $\{\sigma^+, \sigma^+\} = \{\sigma^-, \sigma^-\} = 0$

Check for $\frac{1}{2}$ factors. There may be this factor missing.

This gives

$$H = \sum_{i=1}^{L}\left(\sigma_i^+\sigma_i^- - \frac{1}{2}\right) + \lambda\sum_{i=1}^{L}\left(\sigma_i^+ + \sigma_i^-\right)\left(\sigma_{i+1}^+ + \sigma_{i+1}^-\right)$$

Action of $\sigma^\pm$:

$$\sigma^+|\uparrow\rangle = 0 \qquad \sigma^-|\uparrow\rangle = |\downarrow\rangle \qquad \sigma^z|\uparrow\rangle = |\uparrow\rangle$$
$$\sigma^+|\downarrow\rangle = |\uparrow\rangle \qquad \sigma^-|\downarrow\rangle = 0 \qquad \sigma^z|\downarrow\rangle = -|\downarrow\rangle$$

This representation gives a particle interpretation, If we treat $|\uparrow\rangle = |1\rangle$ and $|\downarrow\rangle = |0\rangle$. We see that $|1\rangle = \sigma^+|0\rangle$ and $|0\rangle = \sigma^-|1\rangle$. Then, the operator e^H gives the dynamics of these particles.

Step 2: The $\sigma_i^\pm$ are almost like fermion operators (at least on the same site).

For fermionic operators,

$$\{f_i^+, f_j^+\} = 0 = \{f_i^-, f_j^-\} \quad \text{and} \quad \{f_i^-, f_j^+\} = \delta_{ij}$$

[for bosons, $\{\,\} \to [\,]$]

The problem is that $\sigma_i^\pm$ on different sites do not anti-commute. This can be corrected by a non-local transformation.

Step 3: Jordan-Wigner transformation.

$$\left.\begin{aligned} f_i^- &= \left(e^{i\pi\sum_{j=1}^{i-1}\sigma_j^+\sigma_j^-}\right)\sigma_i^- \\ f_i^+ &= \sigma_i^+\left(e^{-i\pi\sum_{j=1}^{i-1}\sigma_j^+\sigma_j^-}\right) \end{aligned}\right\}$$ A non-local transformation.

The inverse transformation can be found using $f_i^+ f_i^- = \sigma_i^+\sigma_i^-$ giving

$$\sigma_i^- = \left(e^{-i\pi\sum_{j=1}^{i-1} f_j^+ f_j^-}\right) f_i^- \qquad \sigma_i^+ = f_i^+\left(e^{i\pi\sum_{j=1}^{i-1} f_j^+ f_j^-}\right)$$

You can now check that $f_i^{\pm}$ are fermionic operators

$$\{f_i^-, f_j^+\} = \delta_{ij} \quad \text{and} \quad \{f_i^-, f_j^-\} = 0 = \{f_i^+, f_j^+\} = 0$$

[It takes a bit of steps to check this. See reference uploaded]

In terms of fermionic operators, the Hamiltonian becomes

$$H = \underbrace{2\sum_i f_i^+ f_i^-}_{\text{number operator}} + \lambda \underbrace{\sum_i [f_i^+ - f_i^-][f_{i+1}^+ - f_{i+1}^-]}_{\text{hopping of fermions.}}$$

Step 4:

This is quadratic in fermionic operators. The fermions are non-interactin.
(an interaction needs $f_i^+ f_i^- f_j^+ f_j^-$ terms)

Such a non-interacting Hamiltonian can be easily diagonalized and we get all its eigenvalues.

Fourier transformation.

$$f_j = \frac{1}{\sqrt{2L+1}} \sum_k e^{-ikj} \hat{f}_k \quad \text{with } k = 0, \pm\frac{2\pi}{2L+1}, \pm\frac{2\pi \cdot 2}{2L+1}, \dots, \pm\frac{2\pi n}{2L+1}$$

This gives

$$\boxed{H = 2\sum_{k>0} (1+\lambda\cos k)(f_k^+ f_k^- + f_{-k}^+ f_{-k}^-) + 2\lambda i \sum_{k>0} \sin k\,(f_k^+ f_{-k}^+ + f_k^- f_{-k}^-)}$$

Step 5: This is still not simple enough because of the last term. For thermodynamic limit the highest excited state (or the ground state depending on what sign we choose) is what contributes to free energy. But, because of the last term, the ground state is not simple (NOT the all filled state of f^+f^-).

For this we need to write as

$$H = \sum_k \epsilon_k \, \eta_k^+ \eta_k^-$$

This indeed can be done using Bogoliubov transformation.

$$\eta_K^- = u_K f_K^- + i v_K f_{-K}^+ \qquad \text{and} \qquad \eta_{-K}^- = u_K f_{-K}^- - i v_K f_K^+$$

$$\eta_K^+ = u_K c_K^+ - i v_K c_{-K} \qquad \qquad \eta_{-K}^+ = u_K f_{-K}^- + i v_K f_K^-$$

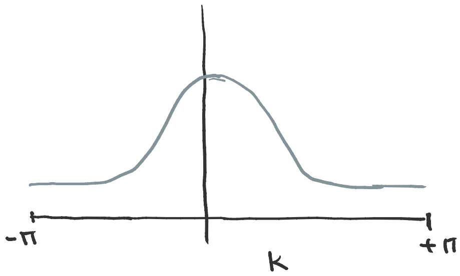
The coefficients u_K, v_K are determined by demanding that

$$\{\eta_K^-, \eta_{K'}^+\} = \delta_{K,K'} \quad \text{and} \quad \{\eta_K^-, \eta_{K'}^-\} = \{\eta_K^+, \eta_{K'}^+\} = 0$$

(such that these are still fermions)

We shall skip this algebra [see page 302 of Mussardo], and arrive at

$$\boxed{H = \sum_K \epsilon_K \eta_K^+ \eta_K^- + \text{constant} \quad \text{where} \quad \epsilon_K = \frac{1}{2}\sqrt{1 + 2\lambda \cos K + \lambda^2}}$$



The largest eigen state corresponds to the case where all K states are filled and the corresponding total energy $\boxed{\Lambda = \sum_K \epsilon_K + \text{constant}}$

Then for large system size

$$Z_{L\times L} \simeq e^{L\Lambda}$$

gives the free energy density

$$f = -\frac{1}{L^2} \log Z_{L\times L} = -\frac{1}{L} \sum_K \epsilon_K - \frac{1}{2} \log(2\sinh 2J)$$

$$= -\frac{1}{2} \log(2\sinh 2J) - \frac{1}{4\pi} \int_{-\pi}^{\pi} dq \, \epsilon(q)$$

For a very detailed description see ch 6.1 of Plischke-Bergersen: equilibrium stat phys.

There are many other approaches to 2d-Ising model.

Heisenberg Spin-chain:

$$H = \sum_i \left[J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z \right]$$

XYZ spin chain.

xxx-spin chain: $J_x = J_y = J_z = J$

$$H = J \sum_i \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} = J \sum_i \{ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z \}$$

$$= J \sum_i \left\{ (f_i^+ - f_i^-)(f_{i+1}^+ - f_{i+1}^-) + \underbrace{\left(f_i^+ f_i^- - \tfrac{1}{2}\right)\left(f_{i+1}^+ f_{i+1}^- - \tfrac{1}{2}\right)}_{\text{Interacting}} \right.$$