Rg for phi 4 theory linear order
Sunday, July 4, 2021 11:57 PM

Perturbative renormalization for ϕ^4 theory

Ref: ch5 of Kardar, vol2
Ch12 of Goldenfeld.
Review of Wilson and Kogut, Phy Rep (1974)

ϕ^4 Action:

$$\mathcal{L}_4 = \int d\bar{r} \left\{ \frac{1}{2}(\nabla\phi)^2 + \frac{t}{2}\phi^2 - h\phi \right\} + \int dr\, u \cdot \phi^4(\bar{r})$$

we shall consider $h=0$, as non-zero h is similar for Gaussian calculation done earlier.

In terms of Fourier modes

$$\mathcal{L}_4 = \int \frac{d\bar{q}}{(2\pi)^d} \left\{ \frac{q^2+t}{2} \right\} |\hat{\phi}(\bar{q})|^2$$

$$+ u \int \frac{d\bar{q}_1}{(2\pi)^d} \frac{d\bar{q}_2}{(2\pi)^d} \frac{d\bar{q}_3}{(2\pi)^d}\, \hat{\phi}(\bar{q}_1)\, \hat{\phi}(\bar{q}_2)\, \hat{\phi}(\bar{q}_3)\, \hat{\phi}(-\bar{q}_1-\bar{q}_2-\bar{q}_3)$$

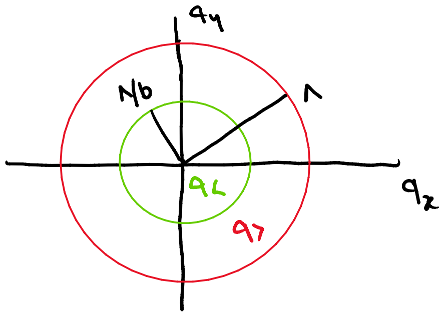
$$= \mathcal{L}_G[\hat{\phi}] + V[\phi]$$

mixes different q-modes.
Difficult to solve exactly.

Setting up perturbation theory: the basic idea.

We want to integrate over $q_> > \Lambda/b$ modes (just like in Gaussian model). However, the ϕ^4 term mixes modes $q_>$ with $q_<$.

$$\mathcal{L}[\phi] = \mathcal{L}_G[\phi_<] + \mathcal{L}_G[\phi_>] + V[\phi_<, \phi_>]$$



[Here for convenience we drop the ^ notation and denote $\hat{\phi}(q) \equiv \phi(q)$
Don't get confused!]

Here $\mathcal{L}_G$ is only Gaussian terms and V denotes quartic terms which involves both $\phi_<$ and $\phi_>$ modes.

Partition function

$$\mathcal{Z} = \int \mathcal{D}[\phi]\, e^{-\mathcal{L}[\phi]}$$

$$= \int \mathcal{D}[\phi_<]\, e^{-\mathcal{L}_<[\phi_<]} \underbrace{\int \mathcal{D}[\phi_>]\, e^{-\mathcal{L}_>[\phi_>] - V[\phi_<,\phi_>]}}_{\text{we want to do this integration}}$$

$$\Downarrow$$

$$\left[\int \mathcal{D}[\phi_>]\, e^{-\mathcal{L}_>}\right] \frac{\int \mathcal{D}[\phi_>]\, e^{-\mathcal{L}_>} \cdot e^{-V}}{\left[\int \mathcal{D}[\phi_>]\, e^{-\mathcal{L}_>}\right]}$$

$$\Downarrow$$

$$Z_0 \langle e^{-V} \rangle_0 [\phi_<]$$

Remember the cumulant expansion

Gives

method in real space RG.

$$\to Z = Z_0 \cdot \int \mathcal{D}[\phi_<]\, e^{-\mathcal{L}_<[\phi_<]} \langle e^{-V} \rangle_0 [\phi_<]$$

Here the subscript 0 denotes averages with respect to $\mathcal{L}_>[\phi_>]$.

Then the new Action

$$\mathcal{L}_{new} = \mathcal{L}_<[\phi_<] - \log \langle e^{-V} \rangle_0 [\phi_<] - \log Z_0$$

cumulant generating function.

constant term does not contribute in the leading singular part of free energy, thus ignored.

Writing in terms of cumulants

$$\mathcal{L}_{new}[\phi_<] = \mathcal{L}_<[\phi_<] + \langle V \rangle_0 [\phi_<] - \frac{1}{2}\left\{ \langle V^2 \rangle_0 - \langle V \rangle_0^2 \right\}[\phi_<] + \cdots$$

After we calculate these cumulants, we still need to do rescaling $q_< = \frac{q}{b}$ and renormalize $\phi_< = z\,\phi$ to keep the $\frac{1}{2}(\nabla\phi)^2$ invariant.

How do we calculate the cumulants :

Sketch of the idea : $\langle \rangle_0$ are the averages computed with Gaussian

$$\mathcal{L}_>(\phi_>) = \int_{\Lambda/b}^{\Lambda} \frac{d\bar{q}_>}{(2\pi)^d} \cdot \frac{t+q^2}{2} \cdot |\phi_>(q_>)|^2$$

The quantity

$$V = u \int \frac{d\bar{q}_1\, d\bar{q}_2\, d\bar{q}_3\, d\bar{q}_4}{(2\pi)^{4d}} \phi(\bar{q}_1)\,\phi(\bar{q}_2)\,\phi(\bar{q}_3)\,\phi(\bar{q}_4) \cdot (2\pi)^d\, \delta(\bar{q}_1+\bar{q}_2+\bar{q}_3+\bar{q}_4)$$

we split by writing

$$\int \frac{d\bar{q}}{(2\pi)^d} \phi(\bar{q}) = \int_0^{\Lambda/b} \frac{d\bar{q}_<}{(2\pi)^d} \phi_<(\bar{q}_<) + \int_{\Lambda/b}^{\Lambda} \frac{dq_>}{(2\pi)^d} \phi_>(\bar{q}_>)$$

This would then generate terms that are products of $\phi_<$ and $\phi_>$. Because $\mathcal{L}_>$ is independant of $\phi_<$, such averages decouple

$$\langle \phi_<(q_<)\, \phi_>(q_>) \rangle_0 = \phi_<(q_<) \langle \phi_>(q_>) \rangle_0$$

Moreover, $\mathcal{L}_>$ is Gaussian and therefore all moments can be computed using Wick's theorem.

Wick's theorem: For a Gaussian theory with

$$\mathcal{L}_G = \int \frac{d\bar{q}}{(2\pi)^d} \cdot r(\bar{q}) \cdot |\phi(\bar{q})|^2$$

the n-th moment

$$\langle \phi(q_1) \cdots \phi(q_n) \rangle_0 = \begin{cases} 0 & \text{for } n = \text{odd} \\ \text{sum over all pairwise contractions} & \text{for } n = \text{even.} \end{cases}$$

More precisely: $\langle \phi(\bar{q}) \rangle_0 = 0$

$$\langle \phi(\bar{q}_1)\, \phi(\bar{q}_2) \rangle_0 = \delta(\bar{q}_1 + \bar{q}_2)\, (2\pi)^d \cdot G(\bar{q}_1)$$

with the propagator $G(\bar{q}) = \dfrac{1}{r(\bar{q})}$

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \langle \phi_1 \phi_2 \rangle \langle \phi_3 \phi_4 \rangle + \langle \phi_1 \phi_3 \rangle \langle \phi_2 \phi_4 \rangle + \langle \phi_1 \phi_4 \rangle \langle \phi_2 \phi_3 \rangle$$

and so on.

The Wick's theorem is a simple consequence of the fact that for Gaussian all cumulants higher than 2 vanish, and the moment generating function

$$\left\langle e^{\sum_{l=1} \lambda_l \phi_l} \right\rangle = e^{\sum_i \sum_j \frac{\lambda_i \lambda_j}{2} \langle \phi_i \phi_j \rangle}$$

Explicit computation of the $\langle V \rangle_0$ term:

As discussed above, the $\langle V \rangle_0$-term involves products of $\phi_<$ and $\phi_>$. There are exactly five possibilities + their permutations.

$$V_0 \sim \phi_< \phi_< \phi_< \phi_<$$

$$V_1 \sim \phi_< \phi_< \phi_< \phi_>$$

$$V_2 \sim \phi_< \phi_< \phi_> \phi_>$$

$$V_3 \sim \phi_< \phi_> \phi_> \phi_>$$

$$V_4 \sim \phi_> \phi_> \phi_> \phi_>$$

(V_1, V_3) $\rightarrow$ 0 under $\langle \rangle_0$ because odd numbers of $\phi_>$

(V_4) $\rightarrow$ independent of $\phi_<$. Thus contributes a constant term to the regular part of free energy density.

Only non-trivial terms are V_0 and V_2.

the V_0 term: this does not involve $\phi_>$ terms, therefore

$$\langle V_0 \rangle_0 = u \int_0^{\Lambda/b} \frac{d\bar{q}_1^< \, d\bar{q}_2^< \, d\bar{q}_3^< \, d\bar{q}_4^<}{(2\pi)^{4d}} \cdot \phi(q_1^<) \phi(q_2^<) \phi(q_3^<) \phi(q_4^<) \, (2\pi)^d \, \delta(\bar{q}_1^< + \bar{q}_2^< + \bar{q}_3^< + \bar{q}_4^<)$$

using $\langle 1 \rangle_0 = 1$.

The $\langle V_2 \rangle_0$ term:

$$\langle V_2\rangle_0 = 6\,u \int_0^{\Lambda/b} \frac{dq_1^<}{(2\pi)^d}\cdot\frac{dq_2^<}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} \frac{dq_3^>}{(2\pi)^d}\cdot\frac{dq_4^>}{(2\pi)^d}\ \phi_<(q_1^<)\,\phi_<(q_2^<)\langle \phi_>(q_3^>)\,\phi_>(q_4^>)\rangle_0$$
$$(2\pi)^d\,\delta\left(q_1^< + q_2^< + q_3^> + q_4^>\right)$$

using $\langle \phi_>(q_3^>)\,\phi_>(q_4^>)\rangle_0 = (2\pi)^d\,\delta\left(q_3^> + q_4^>\right) G(q_3^>)$

gives

$$\langle V_2\rangle_0 = 6\cdot u\cdot \int_0^{\Lambda/b} \frac{dq_1^<\,dq_2^<}{(2\pi)^{2d}}\ \phi_<(q_1^<)\,\phi_<(q_2^<)\ \delta\left(q_1^< + q_2^<\right) \int_{\Lambda/b}^{\Lambda} dq_3^>\cdot G(q_3^>)$$

$$= 6u\cdot\left[\int_{\Lambda/b}^{\Lambda} \frac{dq_3^>}{(2\pi)^d}\,G(q_3^>)\right]\cdot \int_0^{\Lambda/b} \frac{dq_1^<}{(2\pi)^d}\cdot\left|\phi_<(q_1^<)\right|^2$$

Writing all together

Including this in the new $\mathcal{L}$ we write

$$\mathcal{L}_{new} = \int_0^{\Lambda/b} \frac{d\bar{q}_<}{(2\pi)^d}\cdot\left\{\frac{t+q_<^2}{2} + 6u\cdot c_1\right\}\left|\phi_<(\bar{q}_<)\right|^2$$
$$+\ u\int_0^{\Lambda/b} \frac{d\bar{q}_1^<\,d\bar{q}_2^<\,d\bar{q}_3^<\,d\bar{q}_4^<}{(2\pi)^{4d}}\cdot \phi_<(q_1^<)\phi_<(q_2^<)\,\phi_<(q_3^<)\,\phi_<(q_4^<)$$
$$(2\pi)^d\,\delta\left(\bar{q}_1^< + \bar{q}_2^< + \bar{q}_3^< + \bar{q}_4^<\right)$$

where $c_1 = \int_{\Lambda/b}^{\Lambda} \frac{dq_3^>}{(2\pi)^d}\,G(q_3^>)$

We still need to perform two more steps ① rescaling and ② renormalize

① define $q_< = \frac{q}{b}$ and ② $\phi(q) = b^{-\frac{d+2}{2}}\,\phi_<(q_<)$

(to keep the $(\nabla\phi)^2$-term invariant)

Gives

$$\mathcal{L}_{new} = \int_0^{\Lambda} \frac{d\bar{q}}{(2\pi)^d}\cdot\frac{t' + q^2}{2}\cdot\left|\phi(q)\right|^2$$
$$+\ u'\int_0^{\Lambda} \frac{d\bar{q}_1\cdots d\bar{q}_4}{(2\pi)^{4d}}\cdot\phi(\bar{q}_1)\cdots\phi(\bar{q}_4)\cdot(2\pi)^d\,\delta\left(\bar{q}_1+\cdots+\bar{q}_4\right) + \cdots$$

With the RG-transformation

$$\boxed{\begin{aligned} t' &= b^2\left(t + 12 u\, c_1\right) \quad \text{with} \quad c_1 = \int_{\Lambda/b}^{\Lambda} \frac{d\bar q}{(2\pi)^d}\cdot\frac{1}{t+q^2} \\ u' &= b^{-4d}\cdot b^{4\frac{d+2}{2}}\cdot b^d\, u = b^{4-d}\, u \end{aligned}}$$

Remark: see, how for $d>4$, u term decreases under RG flow, confirming that u is irrelevant for Gaussian fixed point.

β-function calculation and RG-flow:

At the n^{th} iteration the formula is similar

$$t_{n+1} = b^2\left(t_n + 12 u_n c_n\right) \quad \text{with} \quad c_n = \int_{\Lambda/b}^{\Lambda} \frac{d\bar q}{(2\pi)^d}\cdot\frac{1}{t_n+q^2}$$

$$u_{n+1} = b^{4-d}\, u_n$$

It is convenient to introduce a parameter τ by $b = e^{d\tau}$ by treating b as small such that $\tau = n\, d\tau$. We also denote $t_n \equiv t(\tau)$, $u_n = u(\tau)$.
This gives,

$$t(\tau + d\tau) = e^{2d\tau}\left[t(\tau) + 12 u(\tau)\int_{\Lambda e^{-d\tau}}^{\Lambda} \frac{dq}{(2\pi)^d}\cdot\frac{1}{t(\tau)+q^2}\right]$$

$$\simeq (1+2d\tau)\left(t(\tau) + 12u\cdot\frac{S_d}{(2\pi)^d}\cdot\frac{\Lambda^d\, d\tau}{t+\Lambda^2}\right)$$

$$\simeq t(\tau) + d\tau\left\{2t(\tau) + 12u\cdot\frac{S_d}{(2\pi)^d}\cdot\frac{\Lambda^d}{t+\Lambda^2}\right\} + \dots$$

here $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area of a unit sphere in d-dimension.

$$\Rightarrow \quad \boxed{\frac{dt}{d\tau} = 2t + 12u\,\frac{S_d}{(2\pi)^d}\cdot\frac{\Lambda^d}{t+\Lambda^2}} = \beta_t(t,u)$$

Similarly

$$\boxed{\frac{du}{d\tau} = (4-d)\,u} = \beta_u(t,u)$$

Fixed point (at this linear order in cumulant expansion)

Evidently $(t=0,\, u=0)$ is a fixed point. This is the Gaussian fixed point.

Eigenvectors of

$$\begin{pmatrix} \frac{\partial\beta_t}{\partial t} & \frac{\partial\beta_t}{\partial u} \\ \frac{\partial\beta_u}{\partial t} & \frac{\partial\beta_u}{\partial u} \end{pmatrix}_{(t=0,u=0)} = \begin{pmatrix} 2 & \frac{12 S_d}{(2\pi)^d}\Lambda^{d-2} \\ 0 & 4-d \end{pmatrix}$$

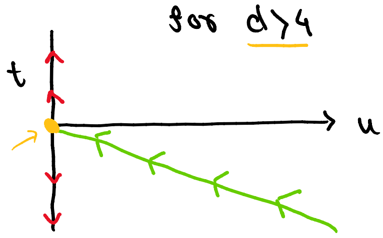
$$\begin{pmatrix} \cdots & \cdots \\ \frac{\partial \beta_u}{\partial t} & \frac{\partial \beta_u}{\partial u} \end{pmatrix}_{(t=0,u=0)} = \begin{pmatrix} \cdots & \cdots \\ 0 & 4-d \end{pmatrix}$$

are the principle directions of RG flow and the eigenvalues y_1, y_2 are the anomalous dimensions.

One principle direction is t with $y_t = 2$

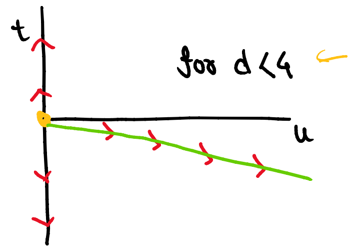
Second principle direction is along $t = -\frac{12 u S_d}{(2\pi)^d} \Lambda^{d-2}$ with $y_2 = 4-d$.

for $d>4$



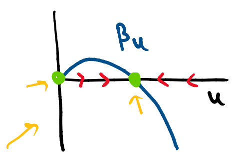
Only one relevant direction (besides h) there fore critical properties governed by the Gaussian fixed point

for $d<4$



The new relevant direction is a critical manifold for another (WF) fixed point which determines the critical properties.

Remark: For $d<4$, the new relevant direction takes to another fixed point. However, to see this new fixed point we need to go to higher order in the cumulant expansion. Naively, we may already see by anticipating



$$\frac{dt}{d\tau} = \beta_t = 2t + 12u \cdot \ast - A u^2$$
$$\frac{du}{d\tau} = \beta_u = (4-d)u - B u^2$$

at second order (we shall see that A, B are positive).

You can see that beside $(t=0, u=0)$ there is an additional fixed point at $(t^* > 0,\ u^* = \frac{4-d}{B})$.

[for $d>4$, $u^* < 0$ is not acceptable as we need u to be positive for the $\mathcal{L}(\phi)$ to be confining, ie, $u\phi^4 > 0$]

In general, this second fixed point is at large u^*.

However, for our perturbation theory we need u to be small, and therefore we can hope to reliably capture the WF-fixed point if u^* is small. This works as long as $4-d = \epsilon$ is small ie we are looking at non integer dimension. This is the idea behind ϵ expansion.

Remark: It turns out that inspite of the requirement for small ϵ, results obtained by simply setting $\epsilon = 1$ are reasonably accurate.

The ϵ expansion is divergent, but assumed to be Borel summable.

Remark: To systematically perform the calculation at higher order, we need a diagramatic representation. (see next lecture note).

Diagramatic (Feynman diagram?) representation

The linear order term $\langle V \rangle_0$: Let's look back at our calculation of the linear order term.

The quantity

$$\langle V \rangle_0 = u \int \frac{d\bar{q}_1\, d\bar{q}_2\, d\bar{q}_3\, d\bar{q}_4}{(2\pi)^{4d}} \left\langle \phi(\bar{q}_1)\, \phi(\bar{q}_2)\, \phi(\bar{q}_3)\, \phi(\bar{q}_4) \right\rangle_0 \cdot (2\pi)^d\, \delta(\bar{q}_1 + \bar{q}_2 + \bar{q}_3 + \bar{q}_4)$$

We decompose the expression by writing

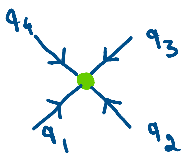
$$\int_0^{\Lambda} \frac{d\bar{q}}{(2\pi)^d}\, \phi(\bar{q}) = \int_0^{\Lambda/b} \frac{d\bar{q}_<}{(2\pi)^d}\, \phi_<(\bar{q}_<) + \int_{\Lambda/b}^{\Lambda} \frac{d\bar{q}_>}{(2\pi)^d}\, \phi_>(\bar{q}_>)$$

Let's look at them term by term.

V_0 term:

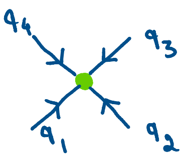
$$\langle V_0 \rangle_0 = \int_0^{\Lambda/b} \frac{d\bar{q}_1^<\, d\bar{q}_2^<\, d\bar{q}_3^<\, d\bar{q}_4^<}{(2\pi)^{4d}} \; u\,(2\pi)^d\, \delta(\bar{q}_1^< + \bar{q}_2^< + \bar{q}_3^< + \bar{q}_4^<)$$

$$\phi_<(q_1^<)\, \phi_<(q_2^<)\, \phi_<(q_3^<)\, \phi_<(q_4^<)$$

$\equiv$ 

[Each arrow represent $\phi_<$ terms. The arrow directions to take care of the $\delta()$ term and $\bullet$ represent the green terms]

Incorporating the additional rescaling and renormalization step we get



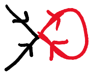
$$= z^4\, b^{-3d} \int_0^{\Lambda} \frac{d\bar{q}_1}{(2\pi)^d} \cdots \frac{d\bar{q}_4}{(2\pi)^d} \cdot u (2\pi)^d\, \delta(\bar{q}_1 + \cdots + \bar{q}_4) \cdot \phi(\bar{q}_1) \cdots \phi(\bar{q}_4)$$

$z = b^{\frac{d+2}{2}}$ came from renormalization of $\phi_<$ field.

V_2 term:

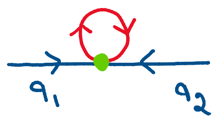
$$\langle V_2\rangle_0 = 6\,u \int_0^{\Lambda/b} \frac{dq_1^<}{(2\pi)^d}\cdot\frac{dq_2^<}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} \frac{dq_3^>}{(2\pi)^d}\cdot\frac{dq_4^>}{(2\pi)^d}\; \phi_<(q_1^<)\,\phi_<(q_2^<)\,\langle \phi_>(q_3^>)\,\phi_>(q_4^>)\rangle_0 \quad \delta(q_3^>+q_4^>)$$

$$(2\pi)^d\,\delta\left(q_1^<+q_2^<+q_3^>+q_4^>\right)$$

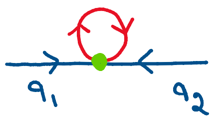


$$= 6\cdot\int_0^{\Lambda/b} \frac{dq_1^<\,dq_2^<}{(2\pi)^{2d}}\;\phi_<(q_1^<)\,\phi_<(q_2^<)\;\int_{\Lambda/b}^{\Lambda}\frac{dq}{(2\pi)^d}\cdot G(q)\;u\,(2\pi)^d\,\delta\left(q_1^<+q_2^<\right)$$

using $\langle\phi_>(q_3^>)\,\phi_>(q_4^>)\rangle_0 = (2\pi)^d\,\delta\left(q_3^>+q_4^>\right)\,G(q_3^>)$

$\equiv 6$ 

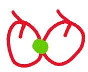
Incorporating the rescaling and renormalization part



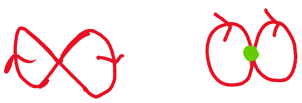
$$= z^2\,b^{-d}\cdot\int_0^{\Lambda}\frac{dq_1\,dq_2}{(2\pi)^{2d}}\;\phi(q_1)\,\phi(q_2)\left[\int_{\Lambda/b}^{\Lambda}\frac{dq}{(2\pi)^d}\cdot G(q)\right]u\,(2\pi)^d\,\delta\left(q_1+q_2\right)$$

The $\langle V_4\rangle_0$ term:

$$\langle V_4\rangle_0 = u\int_{\Lambda/b}^{\Lambda}\frac{dq_1^<}{(2\pi)^d}\cdots\frac{dq_4^>}{(2\pi)^d}\;\langle\phi_<(q_1^<)\,\phi_<(q_2^<)\,\phi_>(q_3^>)\,\phi_>(q_4^>)\rangle_0\;(2\pi)^d\,\delta\left(q_1^<+q_2^<+q_3^>+q_4^>\right)$$

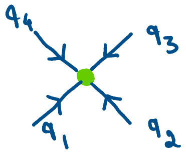
Wick's theorem gives $\langle V_4\rangle = u\,3\left[\int_{\Lambda/b}^{\Lambda}\frac{dq}{(2\pi)^d}\cdot G(q)\right]^2 \equiv 3$ 

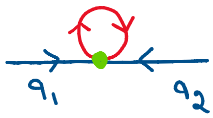
Incorporating rescaling and renormalization

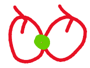


$$= u\left[\int_{\Lambda/b}^{\Lambda}\frac{dq}{(2\pi)^d}\cdot G(q)\right]^2$$

With this above convention for diagrams we express

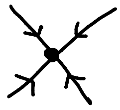
$\langle V \rangle_0 =$ 

$+\ 6$ 

$+\ 3$ 

Any odd open red arm contribute zero amplitud.

Important : note how these diagrams are constructed from a primitive diagram



and then assigning $\phi_>$ or $\phi_<$ to each arm.

For the higher order calculation this would be important.